In defense of linear ocean models

Joe LaCasce
University of Oslo

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The Gulf Stream

Franklin (1770)

Stommel's model

Shallow water equations:

$$\frac{\partial}{\partial t}u + u\frac{\partial}{\partial x}u + v\frac{\partial}{\partial y}u - fv = -g\frac{\partial}{\partial x}\eta + \frac{\tau^x}{\rho H} - \frac{R}{H}u$$

$$\frac{\partial}{\partial t}v + u\frac{\partial}{\partial x}v + v\frac{\partial}{\partial y}v + fu = -g\frac{\partial}{\partial y}\eta + \frac{\tau^y}{\rho H} - \frac{R}{H}v$$

$$\frac{\partial}{\partial t}\eta + \frac{\partial}{\partial x}(u(H+\eta)) + \frac{\partial}{\partial y}(v(H+\eta)) = 0$$

Constant depth, rigid lid:

$$\frac{\partial}{\partial t}\nabla^2\psi + \vec{u}\cdot\nabla(\nabla^2\psi) + \beta\frac{\partial}{\partial x}\psi = \frac{1}{\rho H}\hat{k}\cdot(\nabla\times\vec{\tau}) - r\nabla^2\psi$$

Stommel's model

If the flow is steady and:

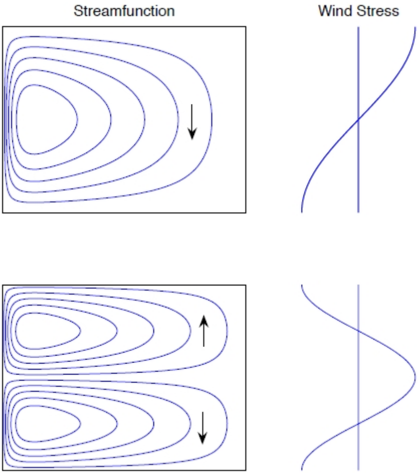
$$\frac{U}{\beta L^2} \ll 1$$

then:

$$\beta \frac{\partial}{\partial x}\psi = \frac{1}{\rho H}\hat{k} \cdot (\nabla \times \vec{\tau}) - r\nabla^2\psi$$

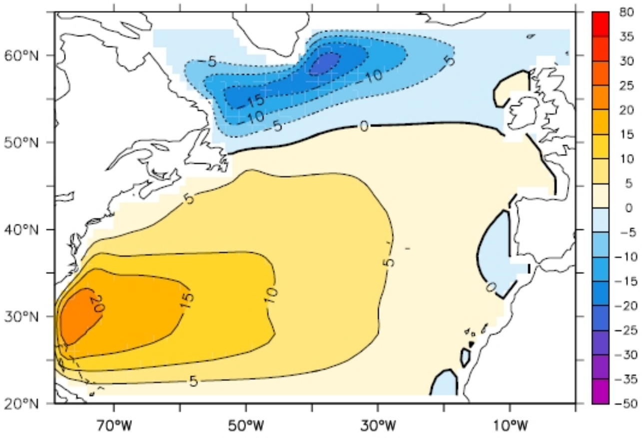
Stommel (1948) solved this in a square basin, using boundary layers (under the assumption that r is small)

Solution



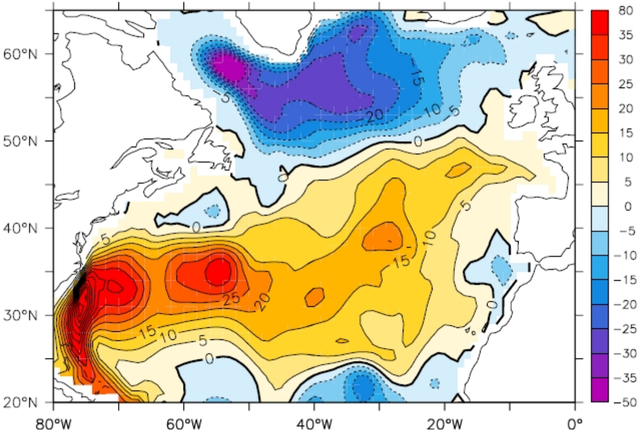
Vallis (2006)

Application to North Atlantic



Vallis (2006)

Application to North Atlantic



Vallis (2006)

The Indian Ocean

De Ruijter's solution

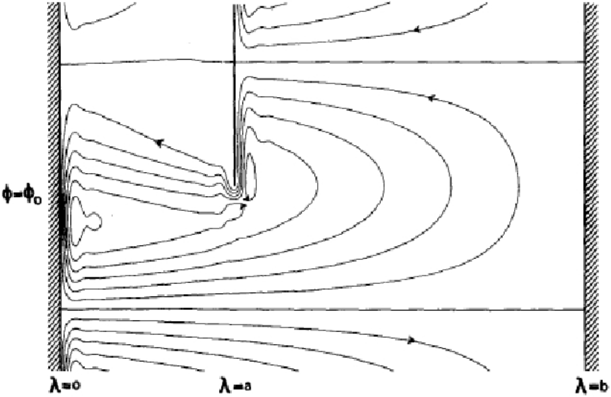
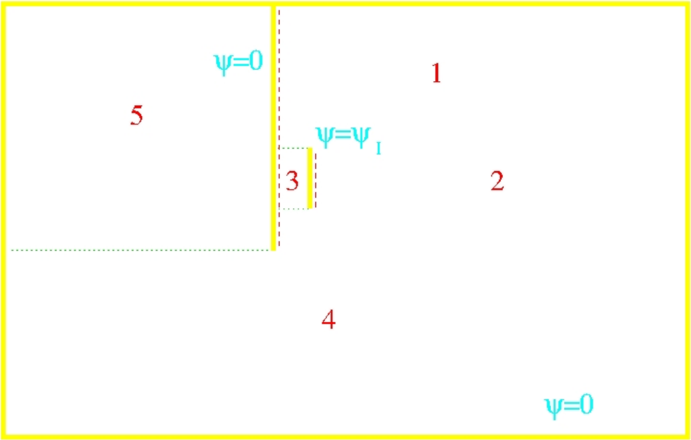


FIG. 3c. As in Fig. 3a, except that curl $\tau = \sin(9\phi + \frac{1}{4}\pi)$, (zeros at −25°, −45°, ---). This wind stress curl resembles (within the sinusoidal approximations with a 40° period) most the actual one. A pronounced free shear layer is now formed in the Atlantic.

De Ruijter (1982)

Madagascar solution



LaCasce and Isachsen (2007)

Madagascar solution

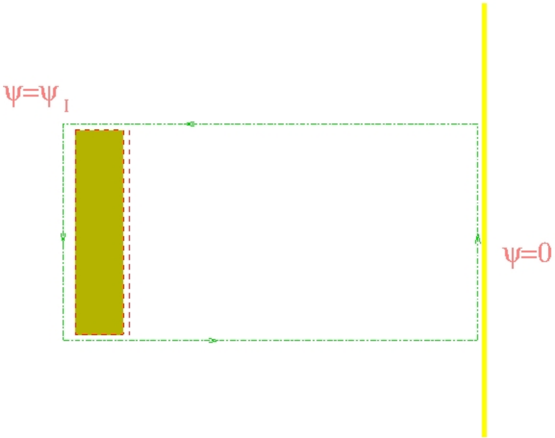
Still based on Stommel's vorticity equation:

$$\beta \frac{\partial}{\partial x}\psi = -\frac{1}{\rho H}\frac{\partial \tau^x}{\partial y} - r\nabla^2\psi$$

But basin geometry results in *discontinuities* in the Sverdrup solution

Also the streamfunction on Madagascar must be determined

Godfrey's Island Rule



Island Rule

Assuming dissipation confined to western boundary currents:

$$\iint \beta \frac{\partial}{\partial x}\psi \, dA = \iint \nabla \cdot (f\vec{u}) \, dA = \frac{1}{\rho H} \iint \nabla \times \vec{\tau} \, dA$$

With Gauss's and Stokes' Laws:

$$\oint f\vec{u} \cdot \hat{n} \, dl = \frac{1}{\rho H} \oint \vec{\tau} \cdot dl$$

If $\tau^x = \tau^x(y)$:

$$\oint \vec{\tau} \cdot dl = [\tau^x(y_N) - \tau^x(y_S)](x_E - x_M)$$

Godfrey (1989)

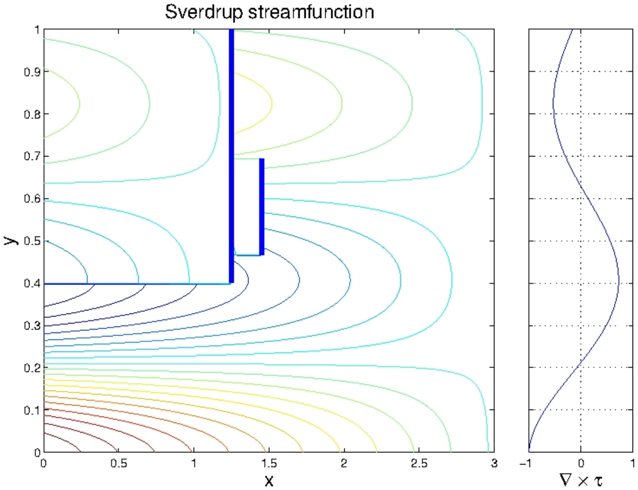
Island Rule

$$\oint f\,\vec{u}\cdot\hat{n}\,dl = f(y_S)\int_{x_I}^{x_E} v\;dx + f(y_N)\int_{x_E}^{x_I} v\;dx$$
$$= [f(y_S) - f(y_N)]\;\psi_I$$

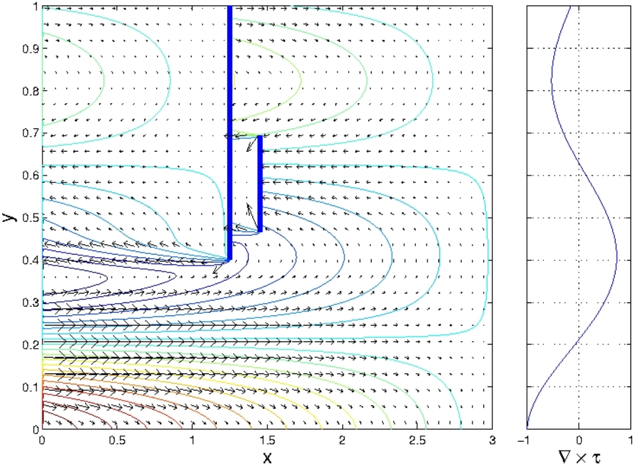
So:

$$\psi_I = \frac{\tau(y_N) - \tau(y_S)}{y_N - y_S}\,(x_E - x_M)$$

Sverdrup streamfunction



Solution with boundary layers



Comparison to observations

F.A. Schott, J.P. McCreary Jr. / Progress in Oceanography 51 (2001) 1–123

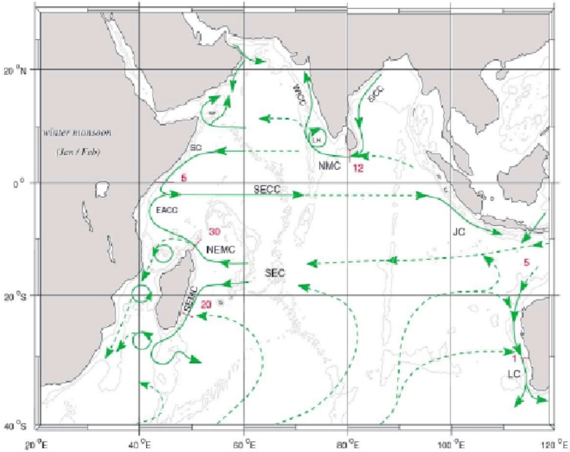
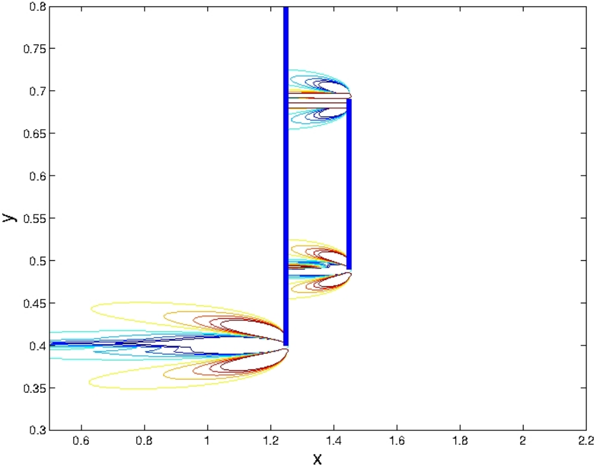
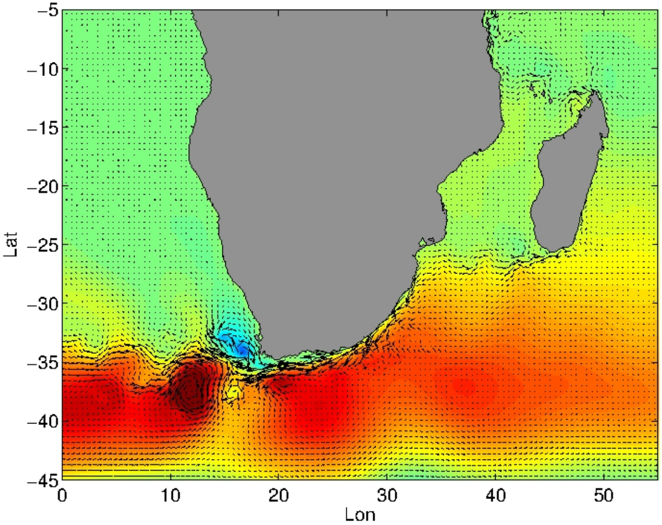


Fig. 9. As in Fig. 8, but for the Northeast Monsoon.

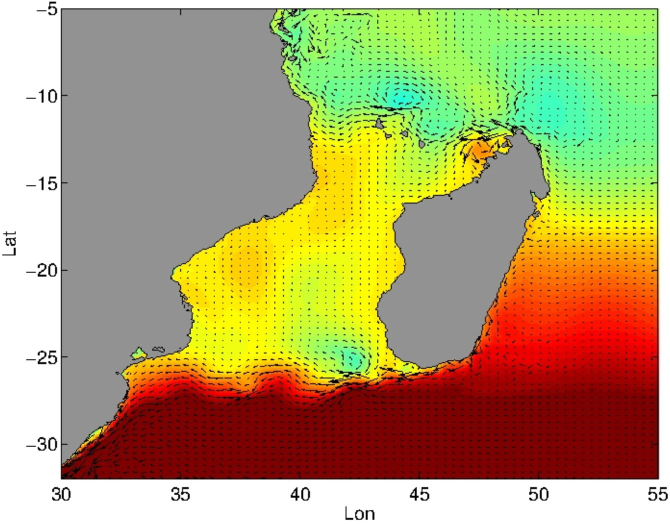
Stability by the Rayleigh-Kuo criterion



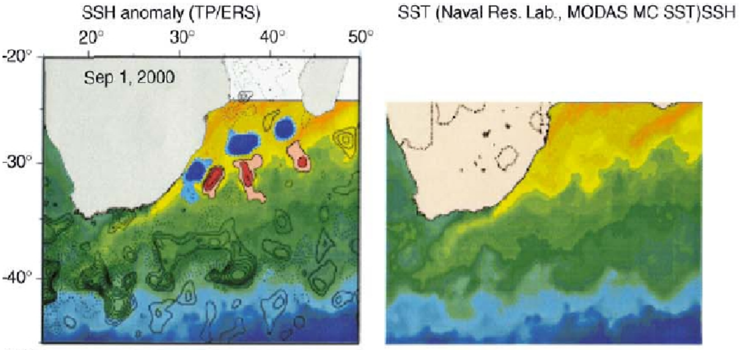
ROMS solution



ROMS solution

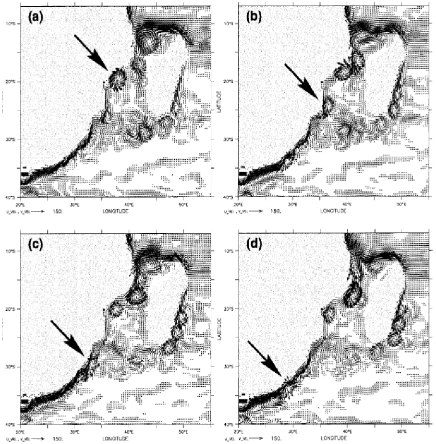


SSH and SST



De Ruijter et al. (2004)

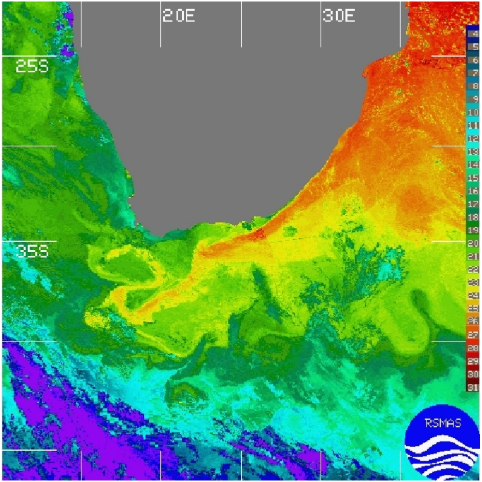
Models



IG. 6. Velocity vectors in 41-m depth (level 3) for model days 30 Aug 31, 20 Oct 31, 19 Nov 31, and 10 Dec 31 (values less than 5 cm s^{-1} are omitted). Note the marked eddy.

Biastoch and Krauss (1999)

Models


20E
30E
25S
35S
4
5
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7
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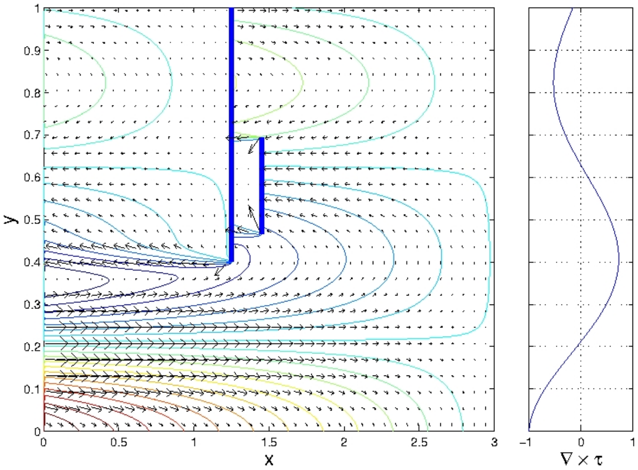
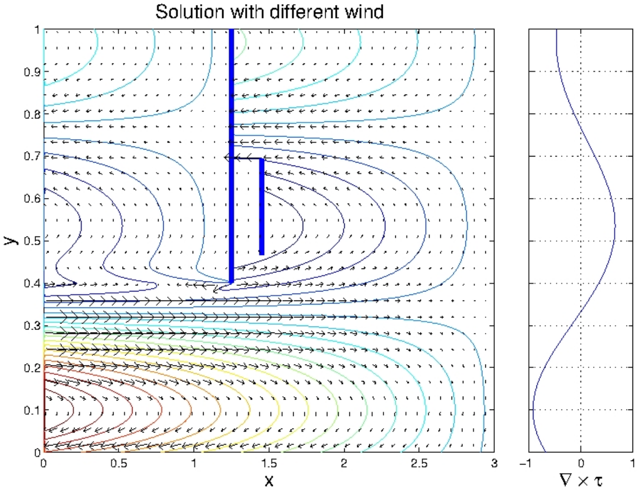
31
RSMAS

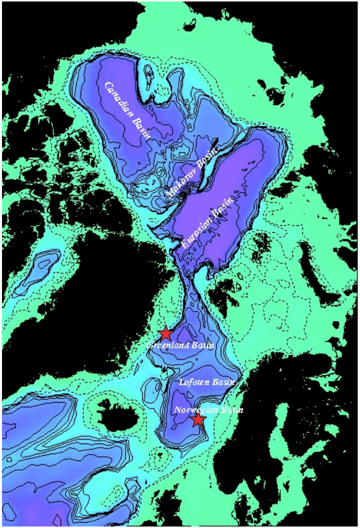
Solution



Different wind forcing



Nordic Seas


Canadian Basin
Makarov Basin
Eurasian Basin
Greenland Basin
Lofoten Basin
Norwegian Basin

Nordic Seas

- Topography cannot be ignored

Shallow water PV equation is:

$$\frac{d}{dt}\frac{\zeta + f}{H} = 0$$

The linear version of this is:

$$\frac{\partial}{\partial t}\zeta + H\vec{u}\cdot\frac{f}{H} = \frac{\partial}{\partial t}\zeta + J(\Psi, \frac{f}{H}) = 0$$

$$Hu = -\frac{\partial}{\partial y}\Psi, \quad Hv = \frac{\partial}{\partial x}\Psi, \quad J(a,b) = \frac{\partial a}{\partial x}\frac{\partial b}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial b}{\partial x}$$

Geostrophic contours

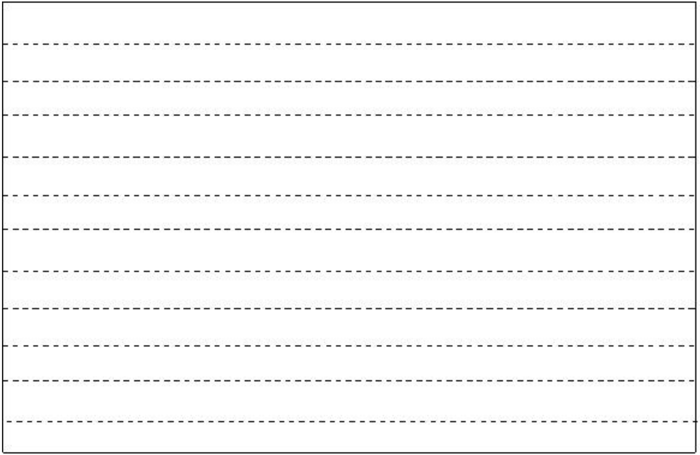
Time-independent flows have:

$$J(\Psi, \frac{f}{H}) = 0$$

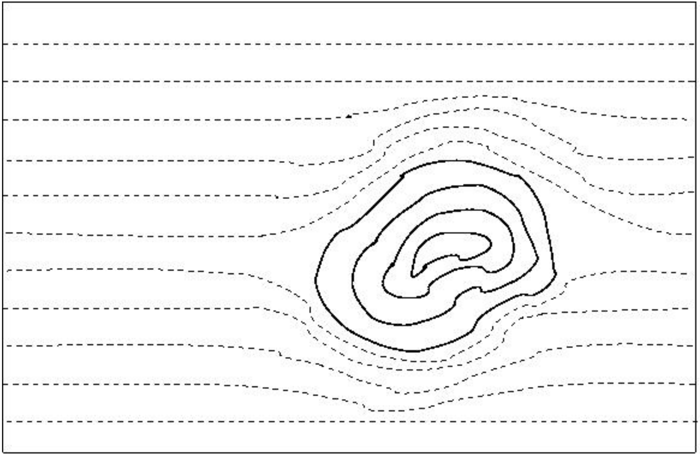
which implies the mean flow is parallel to f/H

- The flow depends on the geometry of the basin

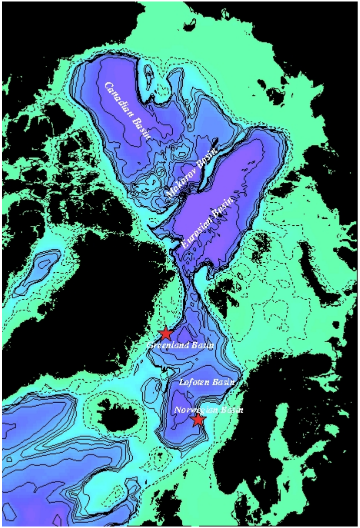
Flat bottom basin



Closed f/H contours



Nordic Seas f/H



Vorticity equation

With forcing and dissipation, the vorticity equation is:

$$\frac{\partial}{\partial t}\zeta + J(\Psi, \frac{f}{H}) = \hat{k} \cdot \nabla \times \frac{\vec{\tau}}{\rho H} - r\zeta$$

Can non-dimensionalize this, assuming weak forcing and weak temporal variations:

$$\epsilon\frac{\partial}{\partial t}\zeta + J(\Psi, \frac{f}{H}) = \epsilon\hat{k} \cdot \nabla \times \frac{\vec{\tau}}{H} - \epsilon\zeta$$

Then expand the streamfunction in ϵ:

$$\Psi = \Psi_0 + \epsilon\Psi_1 + \epsilon^2\Psi_2 + \ \ldots$$

Isachsen et al. 2003

Expansion

The zeroth order flow is parallel to f/H:

$$J(\Psi_0, \frac{f}{H}) = 0$$

The first order terms are:

$$\frac{\partial}{\partial t}\zeta_0 + J(\Psi_1, \frac{f}{H}) = \hat{k} \cdot \nabla \times \frac{\vec{\tau}}{H} - \zeta_0$$

Integrate over a region bounded by an f/H contour:

$$\frac{\partial}{\partial t} \iint \zeta_0 \, dA = \iint \nabla \times \frac{\vec{\tau}}{H} \, dA - \iint \zeta_0 \, dA$$

By Stokes's theorem:

$$\frac{\partial}{\partial t} \oint \vec{u} \cdot \vec{dl} = \oint \frac{\vec{\tau}}{H} \cdot \vec{dl} - \oint \vec{u} \cdot \vec{dl}$$

Expansion

- Determine the circulation on an f/H contour if we have the winds

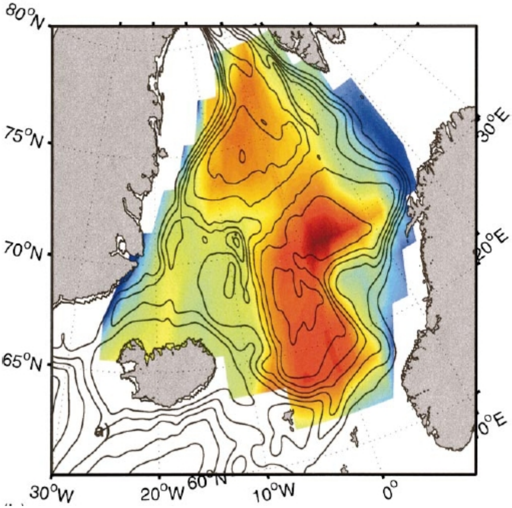
If Fourier transform in time:

$$\vec{u} = \hat{u}(x, y, \omega)e^{i\omega t}, \quad \vec{\tau} = \hat{\tau}(x, y, \omega)e^{i\omega t}$$

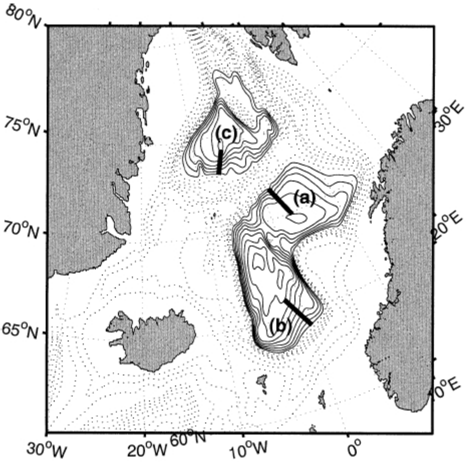
then:

$$\oint \hat{u} \cdot \vec{dl} = \frac{1}{r + i\omega} \oint \frac{\hat{\tau}}{H} \cdot \vec{dl}$$

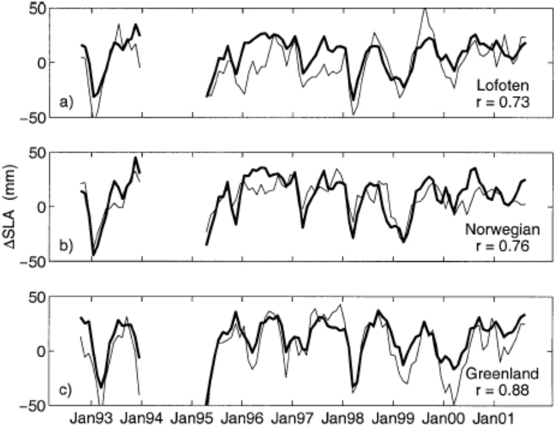
EOF 1



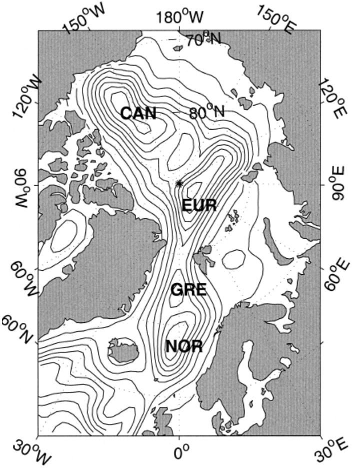
Basin transports



Comparison with SSH



f/H in a GCM


150°W
180°W
150°E
70°N
120°W
120°E
CAN
80°N
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EUR
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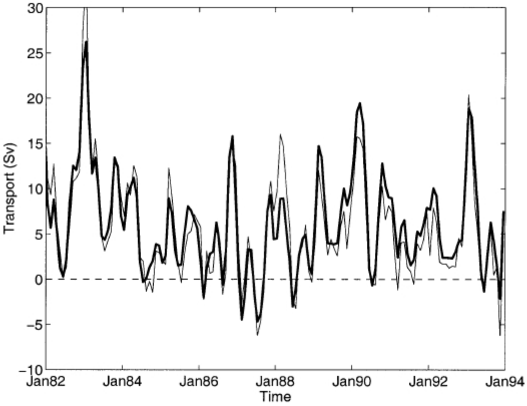
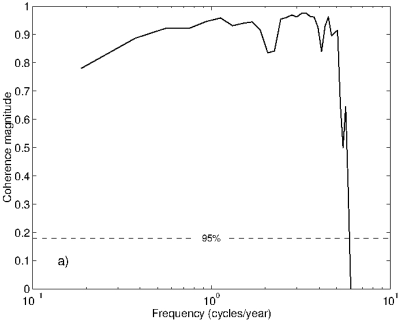
GRE
60°N
NOR
30°W
0°
30°E

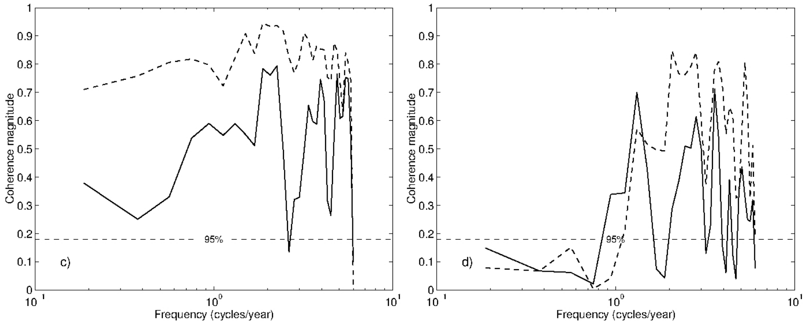
Comparison in the Norwegian basin



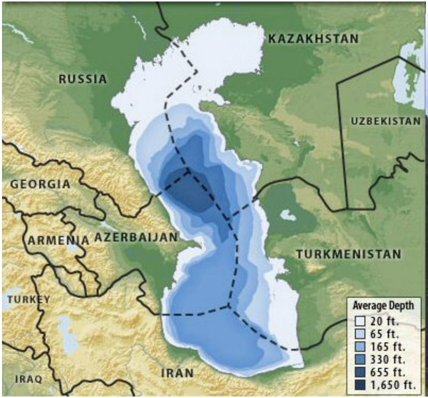
Coherences in Norwegian and Greenland basins



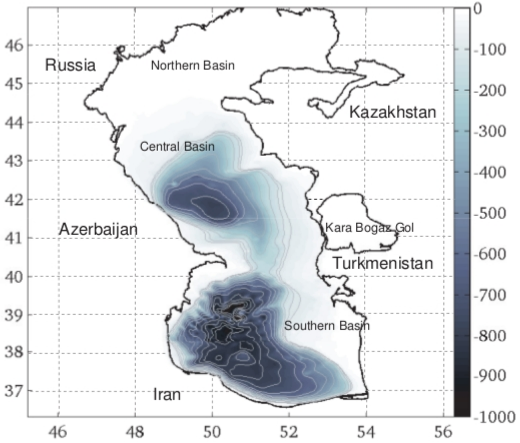
Coherences in Canadian and Eurasian basins



Caspian Sea


KAZAKHSTAN
RUSSIA
UZBEKISTAN
GEORGIA
ARMENIA
AZERBAIJAN
TURKMENISTAN
TURKEY
IRAQ
IRAN
Average Depth
20 ft.
65 ft.
165 ft.
330 ft.
655 ft.
1,650 ft.

Caspian f/H


Russia
Northern Basin
Kazakhstan
Central Basin
Azerbaijan
Kara Bogaz Gol
Turkmenistan
Southern Basin
Iran
46
45
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43
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41
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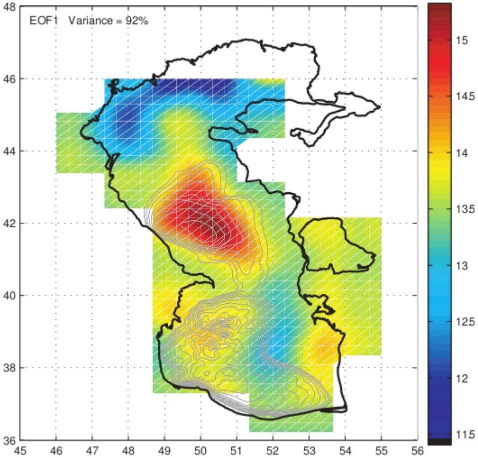
46 48 50 52 54 56
0
-100
-200
-300
-400
-500
-600
-700
-800
-900
-1000

EOF 1


EOF1 Variance = 92%
48
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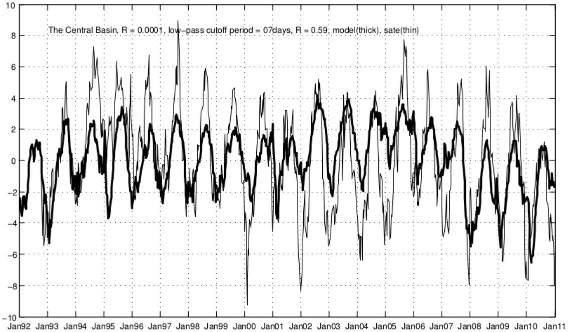
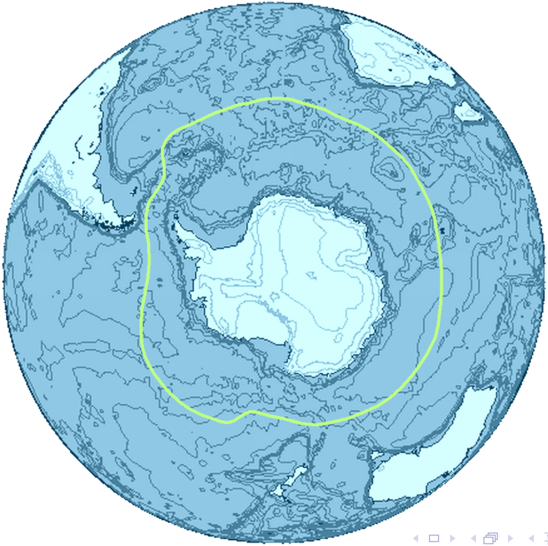
12
115

Transport in the central basin



Ghaffari, Isachsen, LaCasce (2013)

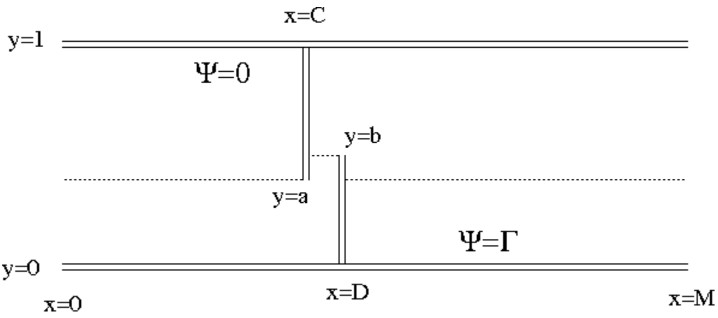
Southern Ocean



Linear models of the ACC

- Stommel (1957) proposed the ACC was in Sverdrup balance
  $\rightarrow$ Not proved though
- Kamenkovich (1962) constructed an "f/H" model of the ACC
  $\rightarrow$ Closed ocean gyres, no ACC
- Gill (1968) studied a "semi-blocked", flat bottom model
  $\rightarrow$ Transport varies as r^{-1}: typically too large (1000 Sv)
- Ishida (1994) proposed a "broken barrier" model

Ishida's model



LaCasce and Isachsen (2010)

Ingredients

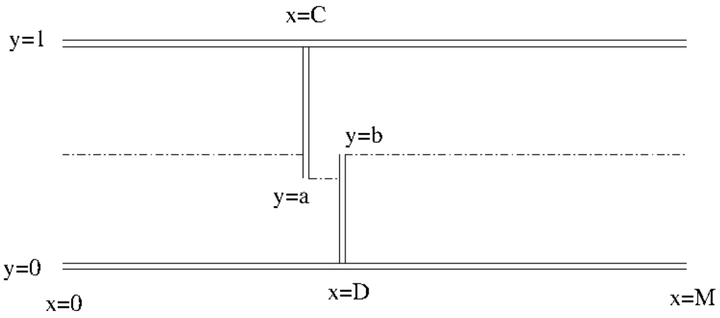
- Stommel vorticity equation:

$$\beta \frac{\partial}{\partial x}\psi = -\frac{1}{\rho H}\frac{\partial \tau^x}{\partial y} - r\nabla^2\psi$$

- Boundary layers on the eastern sides of the barriers and to smooth out the Sverdrup discontinuities in the interior

- Use the Island Rule to determine Γ

Island contour



Island rule

$$\oint f\vec{u}\cdot\hat{n}\,dl = \oint fv\,dx = \frac{1}{\rho H}\oint \tau^x\,dx$$

The left hand side is:

$$f(b)[\psi(C)-\psi(0)] + f(a)[\psi(D)-\psi(C)] + f(b)[\psi(M)-\psi(D)]$$
$$= [f(a)-f(b)]\Gamma$$

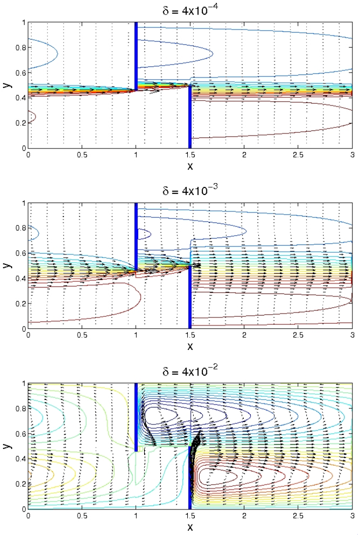
while the right is:

$$= \frac{1}{\rho H}[\tau^x(b)(M+C-D)+\tau^x(a)(D-C)]$$

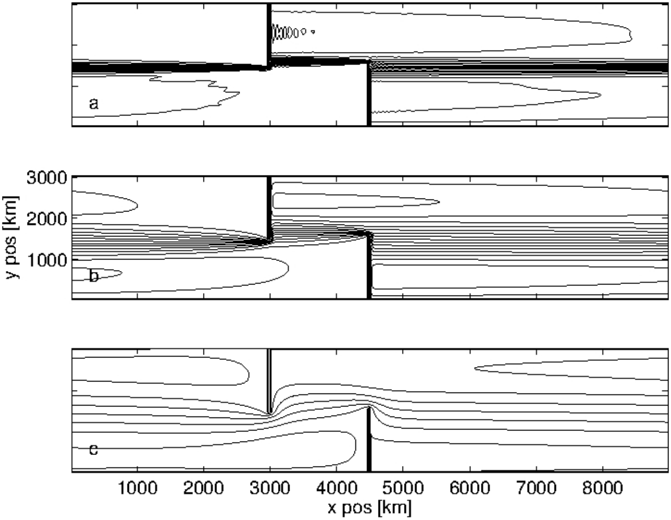
So:

$$\Gamma = \frac{1}{[f(a)-f(b)]\rho H}[M\tau^x(b)+(C-D)(\tau^x(b)-\tau^x(a))]$$

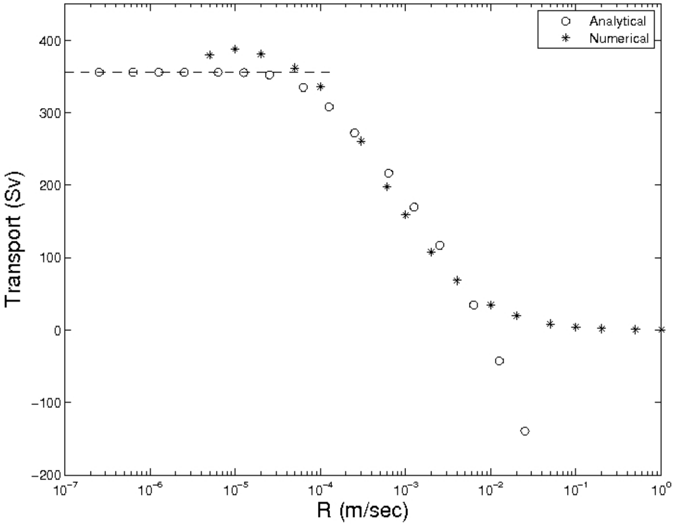
Solutions



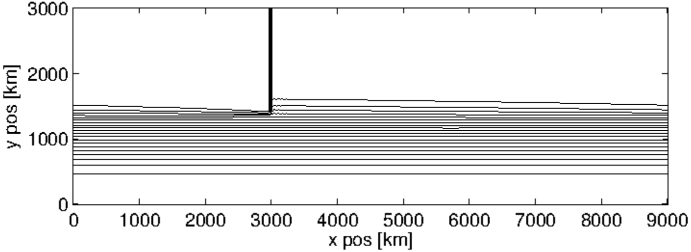
ROMS solution



Transport vs. bottom friction



ROMS Gill solution



Transport $\approx$ 3000 Sv

Equivalent barotropic solutions

Topography is of central importance to the ACC

Can take this into account assuming the current is equivalent barotropic (Killworth, 1992):

$$u(x, y, z) = u(x, y) exp(\frac{z}{z_0})$$

The depth-integrated flow is proportional to (Krupitsky et al., 1996):

$$F \equiv \int_{-H}^{0} exp(\frac{z}{z_0})\, dz = z_0[1 - exp(-\frac{H}{z_0}]$$

Equivalent barotropic solutions

So the in the equivalent barotopic model, f/H is replaced by f/F

If $z_0 \ll H$:

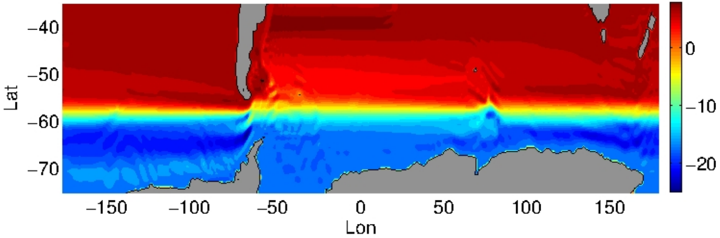
$$f/F \approx f/z_0 \propto f$$

If $z_0 \gg H$:

$$f/F \approx f/H$$

Both limits are recovered

$z_0 = 500$


Lat
-40
-50
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-150
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-50
0
50
100
150
Lon
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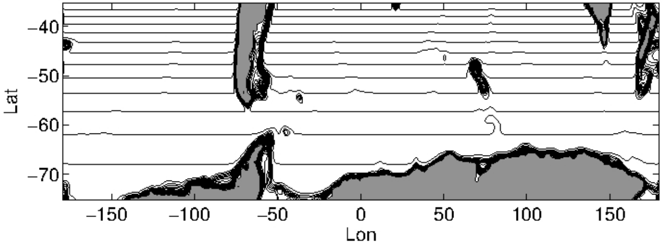
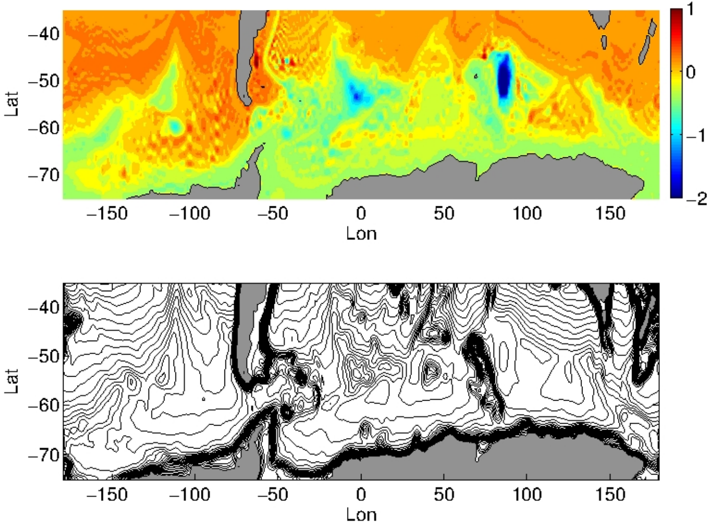
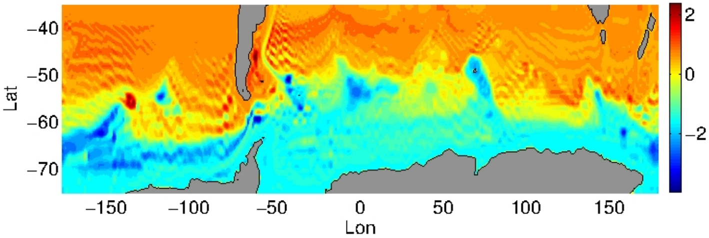
-10
-20

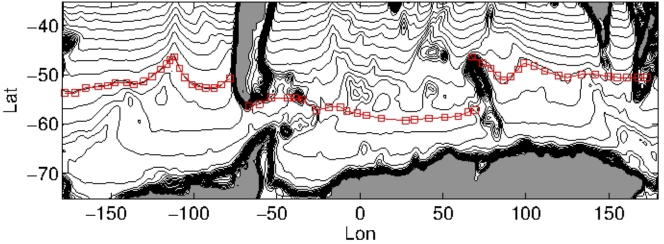

Lat
-40
-50
-60
-70
-150
-100
-50
0
50
100
150
Lon

$z_0 = 3000$

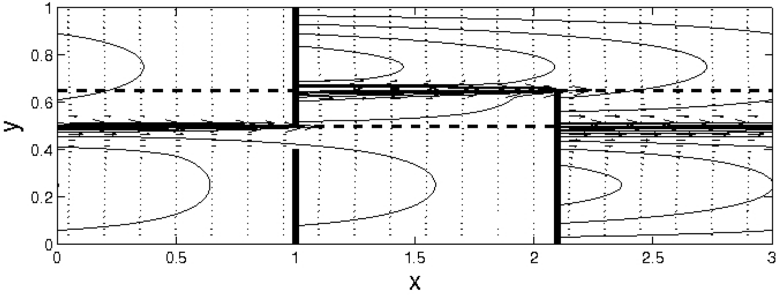

1
0
−1
−2
−40
−50
−60
−70
Lat
−150
−100
−50
0
50
100
150
Lon
−40
−50
−60
−70
Lat
−150
−100
−50
0
50
100
150
Lon

$z_0 = 1400$

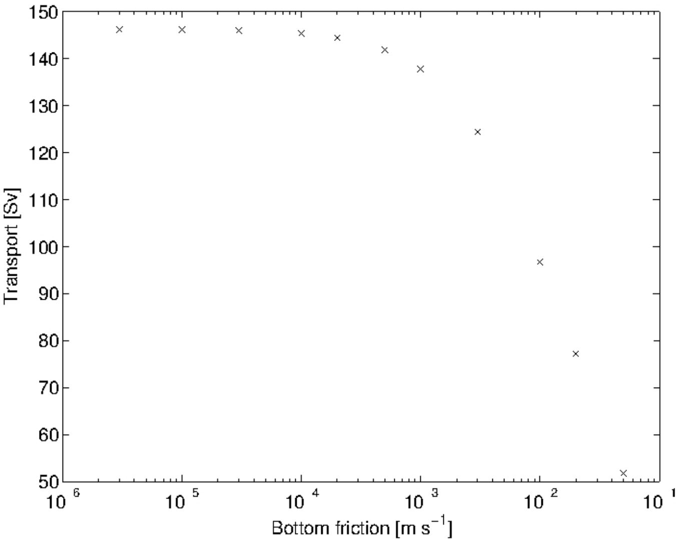

−40
−50
−60
−70
Lat
−150
−100
−50
0
50
100
150
Lon
2
0
−2


−40
−50
−60
−70
Lat
−150
−100
−50
0
50
100
150
Lon

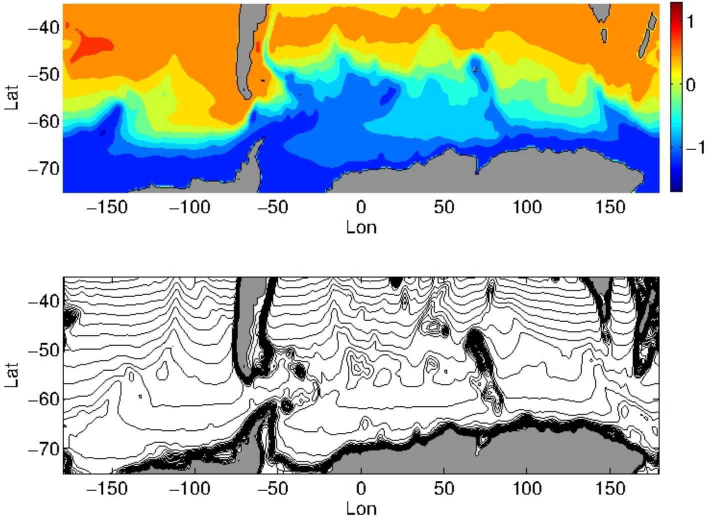
Broken barrier equivalent



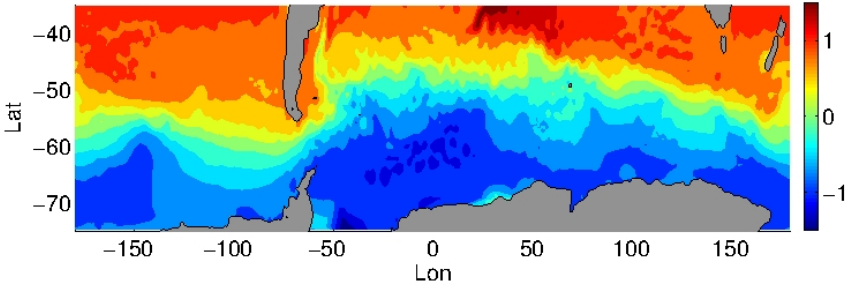
Transport dependence on friction



$z_0 = 1400$, with lateral friction



Mean SSH from satellite



Rio and Hernandez (2004)

Points about the ACC

- Results suggest the ACC has *blocked geostrophic contours*
- Transport is determined by an Island integral of the wind *stress*
- The *form drag balance* is similar to the Island integral, but along the *wrong contour*

Summary

- Linear models are often more than pedagogical
- Can give *quantitative* flow estimates
- No need for long integrations
- No problems with resolution
- Call your program manager today!