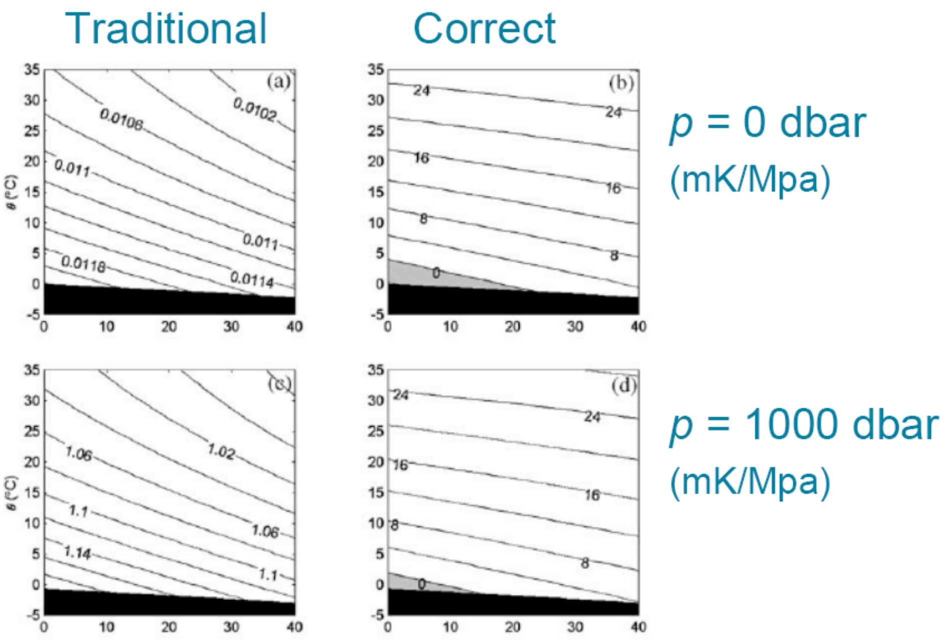
What causes the adiabatic lapse rate?



The adiabatic lapse rate is (a) proportional to the thermal expansion coefficient and (b) is independent of the fluid's compressibility. Indeed, the adiabatic lapse rate changes sign at the temperature of maximum density (where α^t, α^θ and α^Θ all change sign) whereas the compressibility is always positive. This change in sign of the adiabatic lapse rate Γ occurs even though the work done by compression, $\left(p+P_0\right)\mathrm{d}v$, is always positive (for a increase in pressure).

Hence, in cold lakes where the thermal expansion coefficient is negative, the adiabatic lapse rate is negative, so that as the pressure is increased adiabatically, the *in situ* temperature actually decreases! The adiabatic lapse rate Γ represents that change in temperature that is required to keep the entropy (and also θ and Θ) of a seawater parcel constant when its pressure is changed in an adiabatic and isohaline manner.

The traditional explanation has found its way into our textbooks because it works perfectly for a perfect gas; the missing term that we identified just happens to be zero for a perfect gas, but it is the dominant term for a liquid.

Remember, the adiabatic lapse rate has nothing whatsoever to do with the $\left(p+P_0\right)\mathrm{d}v$ work done in changing the internal energy of a fluid parcel. This explanation is wrong even for a perfect gas (where you get the right answer for the wrong reason); for a liquid it is wrong by orders of magnitude.

The adiabatic lapse rate and the potential temperature of ice Ih

Ice Ih is the form of ice with hexagonal packing of the water molecules. This is the form of ice that is found in the range of temperatures and pressures found on planet earth.

The adiabatic lapse rate is equal to the change of *in situ* temperature experienced when pressure is changed while keeping entropy (and salinity) constant. This definition applies separately to both ice and seawater (where one needs to keep not only entropy but also Absolute Salinity constant during the pressure change). In terms of the Gibbs functions of seawater and of ice Ih the adiabatic lapse rates of seawater Γ and of ice Γ^{Ih} are expressed respectively as

$$\Gamma = \left.\frac{\partial t}{\partial P}\right|_{S_{\mathrm{A}},\eta} = \left.\frac{\partial t}{\partial P}\right|_{S_{\mathrm{A}},\Theta} = \left.\frac{\partial t}{\partial P}\right|_{S_{\mathrm{A}},\theta} = -\frac{g_{TP}}{g_{TT}} = \frac{\left(T_0 + t\right)\alpha^t}{\rho\, c_p}, \qquad \text{(Ice\_1)}$$

and

$$\Gamma^{\mathrm{Ih}} = \left.\frac{\partial t}{\partial P}\right|_{\eta} = \left.\frac{\partial t}{\partial P}\right|_{\theta^{\mathrm{Ih}}} = -\frac{g_{TP}^{\mathrm{Ih}}}{g_{TT}^{\mathrm{Ih}}} = \frac{\left(T_0 + t^{\mathrm{Ih}}\right)\alpha^{t\mathrm{Ih}}}{\rho^{\mathrm{Ih}}\, c_p^{\mathrm{Ih}}}\,, \qquad \text{(Ice\_2)}$$

where α^t and $\alpha^{t\mathrm{Ih}}$ are the thermal expansion coefficients of seawater and ice Ih respectively with respect to *in situ* temperature.

The adiabatic lapse rates of seawater and of ice are numerically substantially different from each other. The thermal expansion coefficient of ice does not change sign as does that of seawater when it is cooler than the temperature of maximum density, and the specific heat capacity of ice c_p^{Ih} is only approximately 52% that of seawater c_p.

Figure Ice_1(a) below shows the ratio $\Gamma/\Gamma^{\mathrm{Ih}}$ of the adiabatic lapse rates of seawater and ice at the freezing temperature, as a function of the Absolute Salinity of seawater and pressure. For salinities typical of the open ocean, the ratio $\Gamma/\Gamma^{\mathrm{Ih}}$ is about 0.1 indicating that the *in situ* temperature of ice varies ten times as strongly with pressure when both seawater and ice Ih are subjected to the same isentropic pressure variations. This must be taken into account when considering the vertical motion of frazil ice and the vertical motion of seawater and frazil ice mixtures.

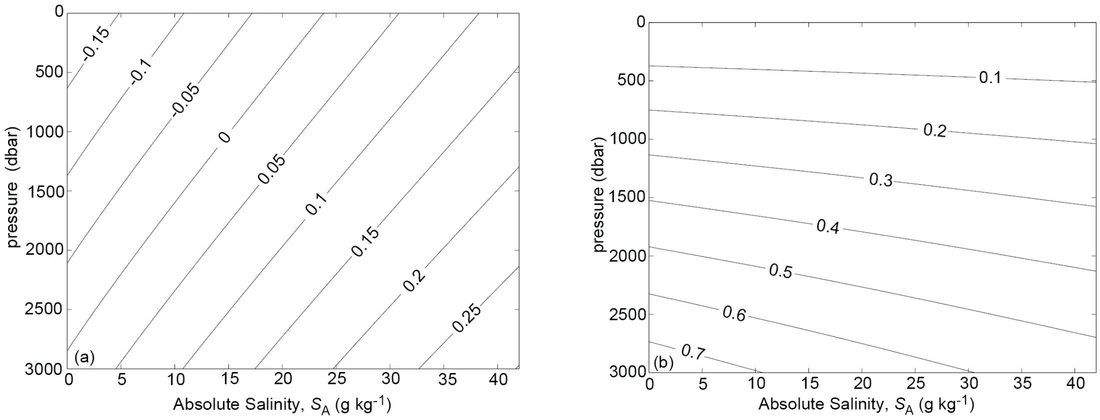


Figure Ice_1. (a) The ratio of the adiabatic lapse rates of seawater and of ice Ih, $\Gamma/\Gamma^{\mathrm{Ih}}$, at the freezing temperature. (b) The difference (in °C) between the potential temperatures of seawater θ and of ice θ^{Ih} for parcels of seawater and ice whose *in situ* temperature is the *in situ* freezing temperature.

The freezing temperature of ice in contact with seawater

The freezing of seawater occurs at the temperature t_{freezing} at which the chemical potential of water in seawater μ^{W} equals the chemical potential of ice μ^{Ih}. Hence the freezing temperature t_{freezing} is found by solving the implicit equation

$$\mu^{\mathrm{W}}\left(S_{\mathrm{A}}, t_{\mathrm{freezing}}, p\right) = \mu^{\mathrm{Ih}}\left(t_{\mathrm{freezing}}, p\right), \qquad \text{(Ice\_3)}$$

or equivalently, in terms of the two Gibbs functions,

$$g\left(S_{\mathrm{A}}, t_{\mathrm{freezing}}, p\right) - S_{\mathrm{A}}\, g_{S_{\mathrm{A}}}\left(S_{\mathrm{A}}, t_{\mathrm{freezing}}, p\right) = g^{\mathrm{Ih}}\left(t_{\mathrm{freezing}}, p\right). \qquad \text{(Ice\_4)}$$

The freezing *in situ* temperatures derived from Eqn. (Ice_4) were converted to the Conservative Temperature at which air-free seawater freezes and are shown in Figure Ice_2(a) as a function of pressure and Absolute Salinity. You can see that whether a water molecule prefers to remain in seawater or prefers to join the solid matrix of water molecules called "ice" depends on the salinity of the seawater and on pressure.

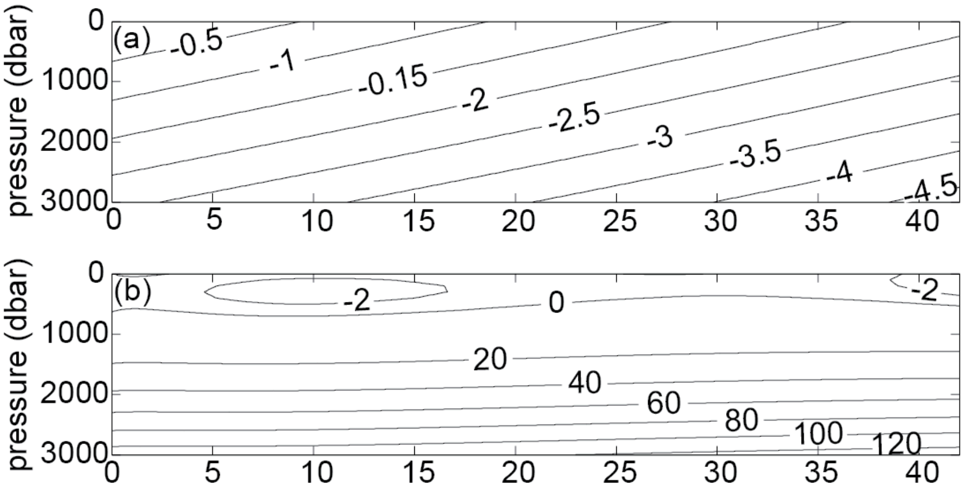


Figure Ice_2. (a) The Conservative Temperature (in °C) at which air-free seawater freezes as a function of pressure and Absolute Salinity. (b) The difference between the freezing Conservative Temperature derived from EOS-80 and that of TEOS-10, with the contours being in mK.

The figure below is another way of plotting the freezing temperature of seawater.

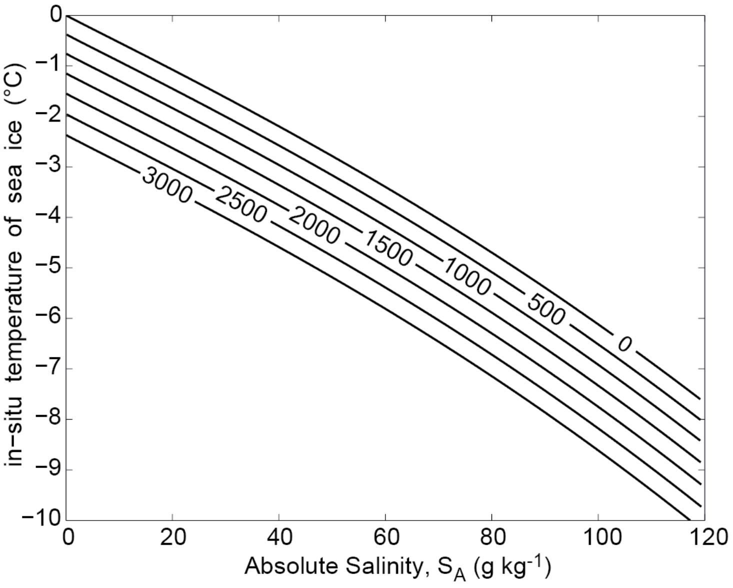


Figure Ice_3. The *in-situ* freezing temperature (in °C) of air-free seawater as a function of pressure (in dbar) and Absolute Salinity, determined from the equilibrium freezing condition Eqn. (Ice_4). In the context of sea ice, the *in situ* temperature is the temperature of both the pure ice Ih phase t^{Ih} and of the trapped pockets of brine.

When discussing the thermodynamic equilibrium between seawater and ice in the oceanographic context there are two common situations. One is called "sea ice" where there are trapped pockets of seawater inside a matrix of ice crystals. This trapped seawater is commonly called "brine" as its salinity can be very large when the temperature is cold.

The other situation is where there are small ice crystals (frazil) are suspended in a much larger volume of seawater so that the mass fraction of ice is small.

In both situations the ice and the seawater exist in thermodynamic equilibrium, so that their *in situ* temperatures are the same. However, as we have seen, the potential temperatures of the ice and seawater phases are different (unless the sea pressure is zero).

Melting of ice into seawater

The First Law of Thermodynamics says that when a process occurs at constant pressure, and without any external input of energy with the environment, then total enthalpy is conserved.

So the conservation equations for mass, salt and enthalpy during an adiabatic melting event at constant pressure are

$$m_{\mathrm{SW}}^{\mathrm{f}} = m_{\mathrm{SW}}^{\mathrm{i}} + m_{\mathrm{Ih}} , \qquad \text{(Ice\_5)}$$

$$m_{\mathrm{SW}}^{\mathrm{f}} S_{\mathrm{A}}^{\mathrm{f}} = m_{\mathrm{SW}}^{\mathrm{i}} S_{\mathrm{A}}^{\mathrm{i}} , \qquad \text{(Ice\_6)}$$

$$m_{\mathrm{SW}}^{\mathrm{f}} h^{\mathrm{f}} = m_{\mathrm{SW}}^{\mathrm{i}} h^{\mathrm{i}} + m_{\mathrm{Ih}} h^{\mathrm{Ih}} . \qquad \text{(Ice\_7)}$$

The superscripts i and f stand for the "initial" and "final" values, that is, the values before and after the melting event, while the subscripts SW and Ih stand for "seawater" and "ice Ih". The mass of ice m_{Ih} is assumed to melt completely, so in the final state there is no ice as all; it is all seawater.

The mass, salinity and enthalpy conservation equations (Ice_5) – (Ice_7) can be combined to give the following expressions for the differences in the Absolute Salinity and the specific enthalpy of the seawater phase due to the melting of the ice,

$$\left(S_{\mathrm{A}}^{\mathrm{f}} - S_{\mathrm{A}}^{\mathrm{i}}\right) = -\frac{m_{\mathrm{Ih}}}{m_{\mathrm{SW}}^{\mathrm{f}}} S_{\mathrm{A}}^{\mathrm{i}} = -\, w^{\mathrm{Ih}} S_{\mathrm{A}}^{\mathrm{i}} , \qquad \text{(Ice\_8)}$$

$$\left(h^{\mathrm{f}} - h^{\mathrm{i}}\right) = -\, w^{\mathrm{Ih}} \left(h^{\mathrm{i}} - h^{\mathrm{Ih}}\right) = \frac{\left(S_{\mathrm{A}}^{\mathrm{f}} - S_{\mathrm{A}}^{\mathrm{i}}\right)}{S_{\mathrm{A}}^{\mathrm{i}}} \left(h^{\mathrm{i}} - h^{\mathrm{Ih}}\right), \qquad \text{(Ice\_9)}$$

where we have defined the mass fraction of ice Ih w^{Ih} as $m_{\mathrm{Ih}} / m_{\mathrm{SW}}^{\mathrm{f}}$. The initial and final values of the specific enthalpy of seawater are given by $h^{\mathrm{i}} = h\left(S_{\mathrm{A}}^{\mathrm{i}}, t^{\mathrm{i}}, p\right) = \hat{h}\left(S_{\mathrm{A}}^{\mathrm{i}}, \Theta^{\mathrm{i}}, p\right)$ and $h^{\mathrm{f}} = h\left(S_{\mathrm{A}}^{\mathrm{f}}, t^{\mathrm{f}}, p\right) = \hat{h}\left(S_{\mathrm{A}}^{\mathrm{f}}, \Theta^{\mathrm{f}}, p\right)$. These equations are illustrated in the following diagram

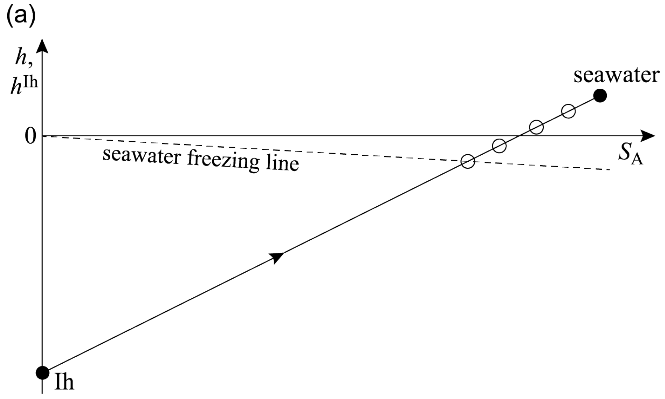


Figure Ice_4(a). This Absolute Salinity – enthalpy diagram illustrates Eqns. (Ice_8) and (Ice_9) which embody the conservation of Absolute Salinity and enthalpy when ice Ih melts into seawater at fixed pressure. The initial values of the Absolute Salinity and enthalpy of seawater and of ice Ih are shown by the two solid dots, and the final values of Absolute Salinity and enthalpy of the seawater after the ice has melted are shown by the four open circles (for four different values of the ice mass fraction w^{Ih}). These final values lie on the straight line on this diagram that connects the initial values (the solid dots).

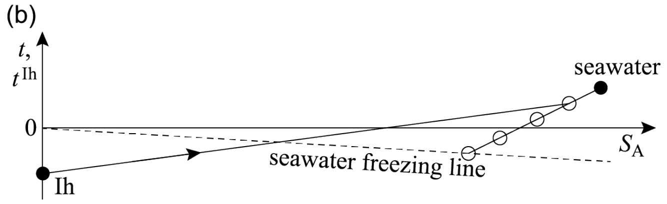


Figure Ice_4(b). The same initial and final data are shown on the Absolute Salinity – *in situ* temperature diagram. Note that the final points (the open circles) do not lie on the straight line connecting the initial points (the solid dots) on this diagram.

The final values of Absolute Salinity, $S_{\mathrm{A}}^{\mathrm{f}}$, and enthalpy, h^{f}, given by Eqns. (8) and (9) are illustrated in Fig. Ice_4(a) for four different values of the ice mass fraction w^{Ih} (the four open circles). These final values, $\left(S_{\mathrm{A}}^{\mathrm{f}}, h^{\mathrm{f}}\right)$, lie on the straight line on the Absolute Salinity - enthalpy diagram connecting $\left(S_{\mathrm{A}}^{\mathrm{i}}, h^{\mathrm{i}}\right)$ and $\left(0, h^{\mathrm{Ih}}\right)$. The fact that the same data does not fall on a straight line on the Absolute Salinity – *in situ* temperature diagram in Fig. Icw_4(b) nicely illustrates that temperature is not conserved when melting occurs.

The linearized expression for the $S_{\mathrm{A}} - \Theta$ ratio when melting occurs

Here we linearize equations (Ice_8) and (Ice_9) to find the expressions (Ice_16) – (Ice_18) for the ratio of the changes in salinity and temperature when a vanishingly small mass fraction of ice melts into seawater at a given pressure.

The enthalpy difference $h^{\mathrm{f}} - h^{\mathrm{i}}$ in Eqn. (Ice_9) is expanded as a Taylor series in the differences in Absolute Salinity and temperature, and the first order terms in these differences are retained, leading to

$$\left(t^{\mathrm{f}} - t^{\mathrm{i}}\right)c_p + \left(S_{\mathrm{A}}^{\mathrm{f}} - S_{\mathrm{A}}^{\mathrm{i}}\right)h_{S_{\mathrm{A}}} \approx \frac{\left(S_{\mathrm{A}}^{\mathrm{f}} - S_{\mathrm{A}}^{\mathrm{i}}\right)}{S_{\mathrm{A}}^{\mathrm{i}}}\left(h^{\mathrm{i}} - h^{\mathrm{Ih}}\right) = -\,w^{\mathrm{Ih}}\left(h^{\mathrm{i}} - h^{\mathrm{Ih}}\right), \qquad \text{(Ice\_10)}$$

where c_p is the specific heat capacity of seawater, $c_p = \partial h/\partial T\big|_{S_{\mathrm{A}},p}$, and $h_{S_{\mathrm{A}}} = \partial h/\partial S_{\mathrm{A}}\big|_{T,p}$ is the derivative of the seawater specific enthalpy with respect to Absolute Salinity at constant *in situ* temperature and constant pressure.

By regarding specific enthalpy to be a function of Conservative Temperature in the functional form $\hat{h}\left(S_{\mathrm{A}},\Theta,p\right)$ the Taylor series expansion of Eqn. (Ice_9) yields

$$\left(\Theta^{\mathrm{f}} - \Theta^{\mathrm{i}}\right)\hat{h}_{\Theta} + \left(S_{\mathrm{A}}^{\mathrm{f}} - S_{\mathrm{A}}^{\mathrm{i}}\right)\hat{h}_{S_{\mathrm{A}}} \approx \frac{\left(S_{\mathrm{A}}^{\mathrm{f}} - S_{\mathrm{A}}^{\mathrm{i}}\right)}{S_{\mathrm{A}}^{\mathrm{i}}}\left(h^{\mathrm{i}} - h^{\mathrm{Ih}}\right) = -\,w^{\mathrm{Ih}}\left(h^{\mathrm{i}} - h^{\mathrm{Ih}}\right), \qquad \text{(Ice\_11)}$$

where $\hat{h}_{\Theta} = \partial h/\partial \Theta\big|_{S_{\mathrm{A}},p}$ is the partial derivative of the seawater specific enthalpy with respect to Conservative Temperature Θ at fixed Absolute Salinity, and $\hat{h}_{S_{\mathrm{A}}} = \partial h/\partial S_{\mathrm{A}}\big|_{\Theta,\,p}$ is the partial derivative of the seawater specific enthalpy with respect to Absolute Salinity at fixed Conservative Temperature Θ. These equations can be rewritten as

$$\delta T\,c_p \;\equiv\; \left(t^{\mathrm{f}} - t^{\mathrm{i}}\right)c_p \;\approx\; \frac{\left(S_{\mathrm{A}}^{\mathrm{f}} - S_{\mathrm{A}}^{\mathrm{i}}\right)}{S_{\mathrm{A}}^{\mathrm{i}}}\left(h - h^{\mathrm{Ih}} - S_{\mathrm{A}}h_{S_{\mathrm{A}}}\right) \;=\; -\,w^{\mathrm{Ih}}\left(h - h^{\mathrm{Ih}} - S_{\mathrm{A}}h_{S_{\mathrm{A}}}\right). \qquad \text{(Ice\_12)}$$

$$\delta\Theta\,\hat{h}_{\Theta} \;\equiv\; \left(\Theta^{\mathrm{f}} - \Theta^{\mathrm{i}}\right)\hat{h}_{\Theta} \;\approx\; \frac{\left(S_{\mathrm{A}}^{\mathrm{f}} - S_{\mathrm{A}}^{\mathrm{i}}\right)}{S_{\mathrm{A}}^{\mathrm{i}}}\left(h - h^{\mathrm{Ih}} - S_{\mathrm{A}}\hat{h}_{S_{\mathrm{A}}}\right) \;=\; -\,w^{\mathrm{Ih}}\left(h - h^{\mathrm{Ih}} - S_{\mathrm{A}}\hat{h}_{S_{\mathrm{A}}}\right). \qquad \text{(Ice\_13)}$$

The bracket on the right-hand side of Eqn. (Ice_12), $h - h^{\mathrm{Ih}} - S_{\mathrm{A}}h_{S_{\mathrm{A}}}$, if evaluated at the freezing temperature $t_{\mathrm{freezing}}\left(S_{\mathrm{A}},p\right)$, is the latent heat of melting (that is, the isobaric melting enthalpy) of ice into seawater. Note that at $p = 0$ dbar $\hat{h}_{S_{\mathrm{A}}}$ is zero while $h_{S_{\mathrm{A}}}$ is nonzero.

The derivation of the isobaric melting enthalpy in Feistel *et al*. (2010) and IOC *et al*. (2010) considered the seawater and ice to be in thermodynamic equilibrium during a slow processes in which heat was supplied to melt the ice while maintaining a state of thermodynamic equilibrium during which the temperature of the combined system changed only because the freezing temperature is a function of the seawater salinity. During this reversible process the enthalpy of the combined system increased due to the heat externally applied. The latent heat of melting is defined to be (from Eqn. (3.34.6) of IOC *et al*. (2010))

$$L_p^{\mathrm{SI}}\left(S_{\mathrm{A}},p\right) \;=\; h\left(S_{\mathrm{A}},t_{\mathrm{freezing}},p\right) - h^{\mathrm{Ih}}\left(t_{\mathrm{freezing}},p\right) - S_{\mathrm{A}}h_{S_{\mathrm{A}}}\left(S_{\mathrm{A}},t_{\mathrm{freezing}},p\right). \qquad \text{(Ice\_14)}$$

In contrast, the present derivation (that is, Eqns. (Ice_12) and (Ice_13)) applies to the common situation when the seawater is warmer than the ice which is melting into it, so that the two phases *are not in thermodynamic equilibrium* with each other during the *irreversible* melting process. That is, the seawater temperature may be larger than its freezing temperature and the ice temperature may or may not be less than its freezing temperature. The guiding thermodynamic principle is that there is no change in the enthalpy of the combined seawater and ice system during the irreversible melting process, since this process occurs adiabatically at constant pressure.

When freezing (as opposed to melting) is considered, the Second Law of Thermodynamics implies that spontaneous freezing cannot occur except when the seawater is at the freezing temperature, and there must be some incremental external change (for example a decrease in pressure in the case of frazil formation, or a loss of heat from the system) in order to induce the freezing.

Taking the limit of melting a small amount of ice into a seawater parcel so that the changes in the seawater temperature and salinity are small, we find from Eqn. (Ice_12) that the ratio of the changes in *in situ* temperature and Absolute Salinity are given by (using Eqn. (Ice_8) for the salinity increment)

$$\begin{aligned} S_{\mathrm{A}}\frac{\delta t}{\delta S_{\mathrm{A}}}\bigg|_{\text{melting at constant p}} &= \frac{h - h^{\mathrm{Ih}} - S_{\mathrm{A}}\,h_{S_{\mathrm{A}}}}{c_p} \\ &= \frac{h\left(S_{\mathrm{A}},t,p\right) - h^{\mathrm{Ih}}\left(t^{\mathrm{Ih}},\,p\right) - S_{\mathrm{A}}\,h_{S_{\mathrm{A}}}\left(S_{\mathrm{A}},t,p\right)}{c_p\left(S_{\mathrm{A}},t,p\right)}\,. \end{aligned} \qquad \text{(Ice\_15)}$$

while the corresponding ratio of the changes in Conservative Temperature and Absolute Salinity are (from Eqn. (Ice_13))

$$\begin{aligned} S_{\mathrm{A}}\frac{\delta\Theta}{\delta S_{\mathrm{A}}}\bigg|_{\text{melting at constant p}} &= \frac{h - h^{\mathrm{Ih}} - S_{\mathrm{A}}\,\hat{h}_{S_{\mathrm{A}}}}{\hat{h}_{\Theta}} \\ &= \frac{\hat{h}\left(S_{\mathrm{A}},\Theta,p\right) - h^{\mathrm{Ih}}\left(t^{\mathrm{Ih}},\,p\right) - S_{\mathrm{A}}\,\hat{h}_{S_{\mathrm{A}}}\left(S_{\mathrm{A}},\Theta,p\right)}{\hat{h}_{\Theta}\left(S_{\mathrm{A}},\Theta,p\right)}, \end{aligned} \qquad \textbf{(Ice\_16)}$$

where the second lines of these equations have been included to be very clear about how these quantities are evaluated. At $p = 0\ \text{dbar}$ these equations become

$$
\begin{aligned}
S_{\mathrm{A}} \frac{\delta\theta}{\delta S_{\mathrm{A}}}\bigg|_{\text{melting at } p=0} &= \frac{h_0 - h_0^{\mathrm{Ih}} - S_{\mathrm{A}}\, h_{S_{\mathrm{A}}}\!\left(S_{\mathrm{A}},\theta,0\right)}{c_p\!\left(S_{\mathrm{A}},\theta,0\right)} \\
&= \frac{h\!\left(S_{\mathrm{A}},\theta,0\right) - h^{\mathrm{Ih}}\!\left(\theta^{\mathrm{Ih}},0\right) - S_{\mathrm{A}}\, h_{S_{\mathrm{A}}}\!\left(S_{\mathrm{A}},\theta,0\right)}{c_p\!\left(S_{\mathrm{A}},\theta,0\right)} ,
\end{aligned}
\qquad \text{(Ice\_17)}
$$

and

$$
S_{\mathrm{A}} \frac{\delta\Theta}{\delta S_{\mathrm{A}}}\bigg|_{\text{melting at } p\,=\,0} = \frac{h_0 - h_0^{\mathrm{Ih}}}{c_p^0} = \Theta - \frac{h^{\mathrm{Ih}}\!\left(\theta^{\mathrm{Ih}},0\right)}{c_p^0} , \qquad \text{(Ice\_18)}
$$

where the potential temperatures of seawater θ and of ice θ^{Ih} are both referenced to $p = 0\ \text{dbar}$. Note that the potential enthalpy of seawater referenced to $p = 0\ \text{dbar}$, $h_0 = h\left(S_{\mathrm{A}},\theta,0\right) = \hat{h}\left(S_{\mathrm{A}},\Theta,0\right)$ is simply c_p^0 times Conservative Temperature Θ where c_p^0 is the constant "specific heat" $c_p^0 \equiv 3991.867\,957\,119\,63\ \ \mathrm{J\,kg^{-1}\,K^{-1}}$.

The use of Conservative Temperature rather than potential temperature means that the slope of the melting process on the $S_{\mathrm{A}} - \Theta$ diagram, $\delta\Theta/\delta S_{\mathrm{A}}$, involves a simpler expression, especially when the melting occurs at the sea surface at $p = 0\ \text{dbar}$, Eqn. (Ice_18), where (i) $\hat{h}_{S_{\mathrm{A}}}\left(S_{\mathrm{A}},\Theta,0\right)$ is zero, and (ii), the relevant "specific heat capacity" of seawater, $\hat{h}_{\Theta} = c_p^0\,\left(T_0 + t\right)/\left(T_0 + \theta\right)$, reduces to the constant c_p^0, so that the specific enthalpy of seawater is simply c_p^0 multiplied by the Conservative Temperature Θ. Note that the numerator of the middle expression of Eqn. (Ice_18) is simply the difference between the potential enthalpies of seawater and of ice.

Note that the right-hand side of Eqn. (Ice_18) is independent of the Absolute Salinity of the seawater into which the ice melts.

We first illustrate these equations for the ratio of the changes of Conservative Temperature to those of Absolute Salinity by considering the melting to occur very close to thermodynamic equilibrium conditions. If both the seawater and the ice were exactly at the freezing temperature at the given values of Absolute Salinity and pressure, then no melting or freezing would occur. In Fig. Ice_5 we consider the limit as the temperatures of both the seawater and the ice approach the freezing temperature. The ratio $\delta\Theta/\delta S_{\mathrm{A}}\big|_{\text{equilibrium}}$ from Eqn. (Ice_16) is shown in Fig. Ice_5(a) with the seawater enthalpy evaluated at the freezing Conservative Temperature and with the ice enthalpy evaluated at the *in situ* freezing temperature, at each value of pressure and Absolute Salinity. This ratio is proportional to the reciprocal of Absolute Salinity, so it is more informative to simply multiply $\delta\Theta/\delta S_{\mathrm{A}}\big|_{\text{equilibrium}}$ by Absolute Salinity S_{A} and this is shown in Fig. Ice_5(b). It is seen that the melting of a given mass of ice into seawater near equilibrium conditions requires between approximately 81 and 83 times as much heat as would be required to raise the temperature of the same mass of seawater by $1°\mathrm{C}$.

The corresponding result for the ratio of the changes of *in situ temperature* and Absolute Salinity near equilibrium conditions $S_{\mathrm{A}}\,\delta t/\delta S_{\mathrm{A}}\big|_{\text{equilibrium}} = L_p^{\mathrm{SI}}\left(S_{\mathrm{A}},p\right)/c_p\left(S_{\mathrm{A}},t_{\text{freezing}},p\right)$ can be calculated from Eqn. (Ice_15), and the difference between $S_{\mathrm{A}}\,\delta t/\delta S_{\mathrm{A}}\big|_{\text{equilibrium}}$ and $S_{\mathrm{A}}\,\delta\Theta/\delta S_{\mathrm{A}}\big|_{\text{equilibrium}}$ is shown in Fig. Ice_5(c). The largest contributor to this difference between Eqns.

(Ice_15) and (Ice_16) is due to the dependence of the specific heat capacity $c_p\left(S_{\mathrm{A}},t_{\mathrm{freezing}},p\right)$ on (i) Absolute Salinity, involving a 6.8% variation over this full range of salinity, and (ii) on pressure, involving a change of 2.2% between 0 dbar to 3000 dbar.

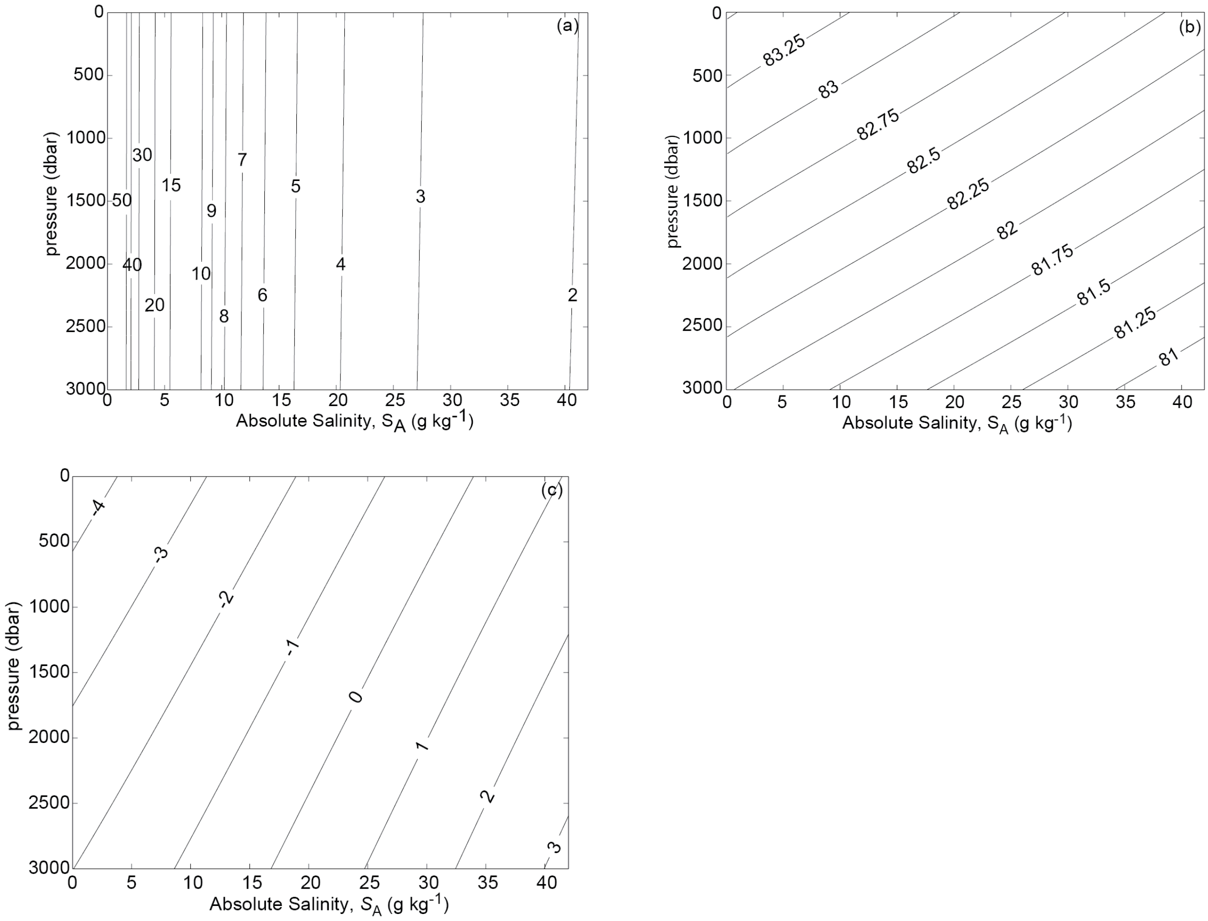


Figure Ice_5. (a) The ratio of the change of Conservative Temperature to that of Absolute Salinity when the melting occurs *very near thermodynamic equilibrium* conditions, $\delta\Theta/\delta S_{\mathrm{A}}\big|_{\mathrm{equilibrium}}$, from Eqn. (Ice_16) with the seawater enthalpy evaluated at the freezing Conservative Temperature and with the ice enthalpy evaluated at the *in situ* freezing temperature, at each value of pressure and Absolute Salinity. The values contoured have units of $\mathrm{K}\left(\mathrm{g\ kg^{-1}}\right)^{-1}$.

(b) This panel is simply Absolute Salinity S_{A} times the values of panel (a), that is, it is the right-hand side of Eqn. (Ice_16), evaluated at equilibrium conditions.

(c) The right-hand side of Eqn. (Ice_15) minus the right-hand side of Eqn. (Ice_16), both evaluated at equilibrium conditions, illustrating the difference between using *in situ* temperature versus Conservative Temperature. The quantities contoured in panels (b) and (c) have units of temperature, K.

Equation (Ice_16) for $S_{\mathrm{A}}\,\delta\Theta/\delta S_{\mathrm{A}}\big|_{\mathrm{melting\ at\ constant\ p}}$ is now illustrated when the seawater and the ice Ih phases are not at the same temperature and *they are not in thermodynamic equilibrium* at the freezing temperature. We begin by considering melting of ice Ih at the sea surface, specifically at $p = 0$ dbar, when Eqn. (Ice_16) reduces to Eqn. (Ice_18), and this equation is illustrated in Fig. Ice_6(a) which applies at all values of Absolute Salinity. The contoured values of Fig. Ice_6(a), $\left(h_0 - h_0^{\mathrm{Ih}}\right)\big/c_p^0 = \Theta - \tilde{h}^{\mathrm{Ih}}\left(\theta^{\mathrm{Ih}}\right)\big/c_p^0$, increase as 1.0 times changes in Θ and decrease approximately as $c_p^{\mathrm{Ih}} / c_p^0 \approx 0.52$ times changes in the temperature of the ice.

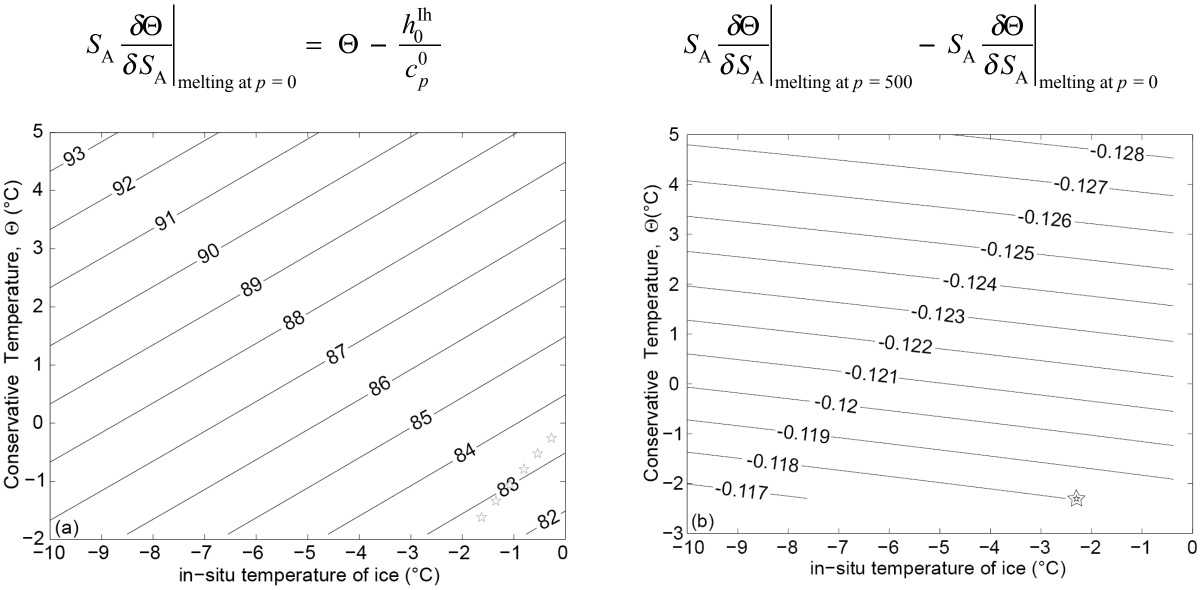


Figure Ice_6. (a) Contours of Eqn. (Ice_18), $S_{\mathrm{A}}\,\delta\Theta/\delta S_{\mathrm{A}}\big|_{\text{melting at } p\,=\,0} = \left(h_0 - h_0^{\mathrm{Ih}}\right)/c_p^0 = \Theta - \tilde{h}^{\mathrm{Ih}}\left(\theta^{\mathrm{Ih}}\right)/c_p^0$, for the melting of ice Ih into seawater at $p = 0\,\mathrm{dbar}$. The six stars are at the freezing temperatures (t and Θ) for Absolute Salinity values starting at $5\,\mathrm{g\,kg^{-1}}$ with increments of $5\,\mathrm{g\,kg^{-1}}$ up to $30\,\mathrm{g\,kg^{-1}}$. (b) Difference between contours of Eqn. (16) at $p = 500\,\mathrm{dbar}$, $S_{\mathrm{A}}\,\delta\Theta/\delta S_{\mathrm{A}}\big|_{\text{melting at } p\,=\,500}$, and the corresponding ratio of panel (a) (where the pressure was $0\,\mathrm{dbar}$) at $S_{\mathrm{A}}^{\mathrm{i}} = S_{\mathrm{SO}} = 35.16504\ \mathrm{g\ kg^{-1}}$. The double-starred point is at the freezing temperatures (t and Θ) at $p = 500\,\mathrm{dbar}$ and $S_{\mathrm{A}}^{\mathrm{i}} = S_{\mathrm{SO}} = 35.16504\ \mathrm{g\ kg^{-1}}$.

Taking the potential enthalpy of ice Ih to be a conservative variable

By comparing panels (a) and (b) of Fig. Ice_6 we are able to deduce a very important approximation that will prove invaluable to coupled ocean/ice modelling. Panel (b) shows the error in assuming that it is the potential enthalpy of ice that is conserved when ice melts into seawater, rather than taking the enthalpy of ice to be conserved, which is the correct thing to do. At a pressure of 500 dbar the assumption that the sum of the *potential* enthalpies of ice and seawater are conserved leads to an error of 0.15% in the change in Conservative Temperature of the seawater as a result of melting. Most of this error is due to the assumption regarding ice, not seawater, since the error involved with assuming that the Conservative Temperature of seawater is totally conservative reaches a maximum of 0.15% only at a much larger pressure of 4000 dbar (Graham and McDougall, 2013).

The ratio of Eqns. (Ice_16) to (Ice_18) is

$$\frac{\left.\dfrac{\delta\Theta}{\delta S_{\mathrm{A}}}\right|_{\text{melting at constant p}}}{\left.\dfrac{\delta\Theta}{\delta S_{\mathrm{A}}}\right|_{\text{melting at } p\,=\,0}} = \frac{\left(T_0+\theta\right)}{\left(T_0+t\right)}\left[\frac{h-h^{\mathrm{Ih}}-S_{\mathrm{A}}\,\hat{h}_{S_{\mathrm{A}}}}{h_0-h_0^{\mathrm{Ih}}}\right]$$

$$= 1+\frac{\left(\theta-t\right)}{\left(T_0+t\right)}+\frac{\left(T_0+\theta\right)}{\left(T_0+t\right)}\frac{\left[\left(h-h_0\right)-\left(h^{\mathrm{Ih}}-h_0^{\mathrm{Ih}}\right)-S_{\mathrm{A}}\,\hat{h}_{S_{\mathrm{A}}}\right]}{\left(h_0-h_0^{\mathrm{Ih}}\right)}, \qquad \text{(Ice\_19)}$$

and the combination of enthalpy differences in the numerator of the last term can be expressed as

$$\left[h-h_0-\left(h^{\mathrm{Ih}}-h_0^{\mathrm{Ih}}\right)-S_{\mathrm{A}}\,\hat{h}_{S_{\mathrm{A}}}\right] = \int_{P_0}^{P}\left[\hat{v}\left(S_{\mathrm{A}},\Theta,p'\right)-\tilde{v}^{\mathrm{Ih}}\left(\theta^{\mathrm{Ih}},p'\right)\right]dP' - S_{\mathrm{A}}\int_{P_0}^{P}\hat{v}_{S_{\mathrm{A}}}\left(S_{\mathrm{A}},\Theta,p'\right)dP'. \qquad \text{(Ice\_20)}$$

The last term here is small, showing that the dominant contribution is simply the pressure integral of the difference in the specific volumes of seawater and of ice.

In Eqn. (Ice_19) the second term on the right-hand side, $\left(\theta-t\right)/\left(T_0+t\right)$, is small compared with the third, so that the non-unity nature of Eqn. (Ice_19) can be understood as being due to this third term, evaluated with the aid of Eqn. (Ice_20), and this evaluation agrees with the plot of Fig. Ice_6(b).

We will take advantage of the smallness of panel (b) versus panel (a) of Fig. Ice_6, or equivalently, the fact that Eqn. (Ice_19) is quite close to unity, to treat the potential enthalpy of ice as conserved during not only advection but also during melting and freezing events. This will greatly reduce the complexity of coupled ocean/ice numerical models. This approximation brings the same simplicity to ice as the introduction of Conservative Temperature has brought to physical oceanography, in that the only variables that now need to be considered when discussing "heat" budgets of seawater and of ice are the potential enthalpies of seawater and of ice.

An illustration from the Amery Ice Shelf

Figure Ice_7 shows oceanographic data obtained under the Amery Ice Shelf that illustrates the ratio of the changes in Absolute Salinity and Conservative Temperature, as given by Eqn. (Ice_16), when melting of ice occurs. The vertical profile named AM06 begins under the ice at a pressure of 546 dbar and the uppermost 175 of the vertical profile is shown. The data in the uppermost 50-100 dbar is closely aligned with the ratio given by Eqn. (Ice_16) (as shown by the dashed line) evaluated at this pressure and with the ice temperature being the freezing temperature at this salinity and pressure. Two freezing lines are shown in Fig. Ice_7(b), for pressures of 0 dbar and 578 dbar.

Any observations cooler than the freezing temperature appropriate to 0 dbar is evidence of the influence of melting of ice or of heat lost by conduction through the ice. AM06 is located on the eastern side of the ice shelf in an area that is melting, as can be inferred by the presence of ocean water at AM06 that is well above the *in situ* freezing temperature at the base of the ice shelf. This water is thought to be flowing in a primarily southwards direction from the open ocean as it enters the under-ice cavity. The other CTD profile was taken from borehole AM05, located on the western side of the ice shelf in an area that is refreezing (as is drawn in panel (a)) and represents flow that has likely come from deeper in the sub-ice-shelf cavity, than at AM06 (Post *et al*., 2013) and hence has been in contact with the ice for longer. The upper 50m or so of this cast is at the freezing temperature of seawater at this pressure. For both casts the data near the upper part of the water column has the ratio of the changes of S_{A} and Θ in close agreement to the ratio given by Eqn. (Ice_16), the ratio predicted from melting ice into seawater (dashed lines). The ice temperature that is needed to calculate this $S_{\mathrm{A}} - \Theta$ ratio for each location has been taken to be the *in situ* freezing temperature of ice in contact with the seawater at the pressure at the base of the ice shelf. Moreover, on this figure the uppermost 100m of data of the AM05 data is approximately related to that of the AM06 data through the $S_{\mathrm{A}} - \Theta$ ratio of Eqn. (Ice_16). This would be consistent with the notion that the same fluid is proceeding from AM06 to AM05 without being exposed to significant heat loss Q to the ice (see panel (a)). The vertical profiles shown in panel (b) are the average of several vertical profiles taken over the course of two days, and the two locations were drilled within two weeks of each other.

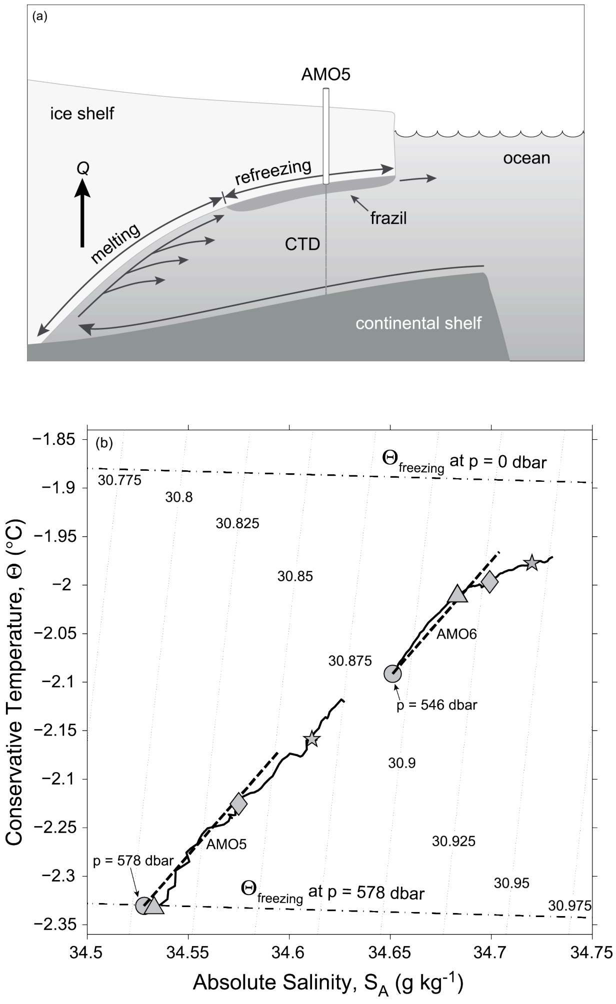


Figure Ice_7. (a) Sketch of the flow under an ice shelf. An inflow of relatively warm water from the open ocean provides heat to melt the ice shelf. Buoyant freshwater that is released during the melting process rises along the underside of the ice shelf and can become locally supercooled at a shallower depth, leading to the formation of frazil and basal accretion of marine ice.

(b) The top 175m of two CTD profiles taken below the Amery Ice Shelf in East Antarctica at a melt site and at a refreeze site are shown. The warmer and saltier of the two casts is AM06 (see Fig 1 of Galton-Fenzi *et al*. (2012)) starting at a pressure of 546 dbar. The large round dot is ocean data very near the ice at 546 dbar, the triangle is 50 dbar deeper, the diamond 100 dbar deeper and the star is 150 dbar below the bottom of the ice shelf at this location, indicated by the circle. The other vertical cast, AM05, is typical of re-freezing locations. The uppermost 50 dbar of this cast is all at the freezing temperature at this pressure.

Melting of sea ice into seawater

Sea ice contains a certain mass fraction of brine trapped inside the ice matrix. Sea ice is produced when the surface of the ocean is cooled rapidly by very cool air. The ice crystals then form so fast that some of the seawater is trapped in small "pockets" inside the matrix of ice crystals.

We can quantify the melting of sea ice into seawater by conserving the same three quantities, namely (i) mass, (ii) salt, and (iii) enthalpy, leading to

> (see McDougall, T. J., P. M. Barker, R. Feistel and B. K. Galton-Fenzi, 2014: Melting of ice and sea ice into seawater, and frazil ice formation. *Journal of Physical Oceanography*, **44**, 1751-1775. for details)

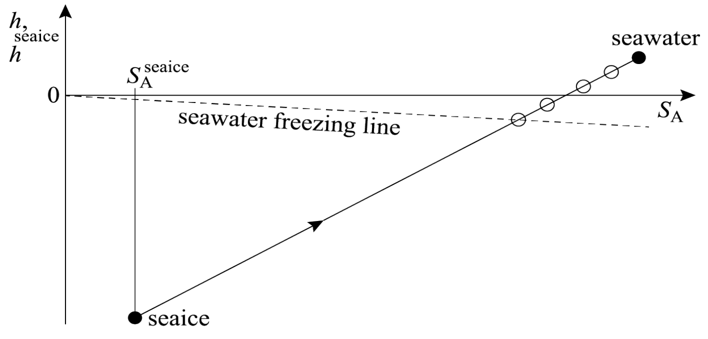
$$\left(S_{\mathrm{A}}^{\mathrm{f}} - S_{\mathrm{A}}^{\mathrm{i}}\right) \;=\; -\,\frac{m_{\mathrm{seaice}}}{m_{\mathrm{SW}}^{\mathrm{f}}}\left(S_{\mathrm{A}}^{\mathrm{i}} - S_{\mathrm{A}}^{\mathrm{seaice}}\right) \;=\; -\,w^{\mathrm{seaice}}\left(S_{\mathrm{A}}^{\mathrm{i}} - S_{\mathrm{A}}^{\mathrm{seaice}}\right)$$

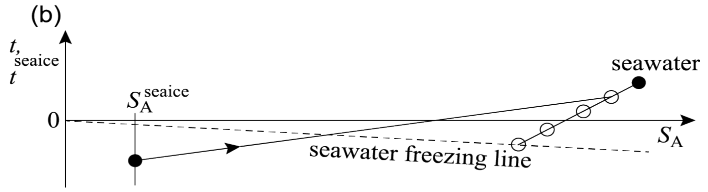
$$\begin{aligned}\left(h^{\mathrm{f}} - h^{\mathrm{i}}\right) \;&=\; -\,w^{\mathrm{seaice}}\left(h^{\mathrm{i}} - h^{\mathrm{Ih}}\right) \;+\; w^{\mathrm{seaice}}\,\frac{m_{\mathrm{brine}}}{m_{\mathrm{seaice}}}\left(h^{\mathrm{brine}} - h^{\mathrm{Ih}}\right)\\ &=\; -\,w^{\mathrm{seaice}}\left(h^{\mathrm{i}} - h^{\mathrm{seaice}}\right).\end{aligned}$$

where the specific enthalpy of the composite material "seaice" is defined as the mass-weighted sum of the specific enthalpies of the two phases,

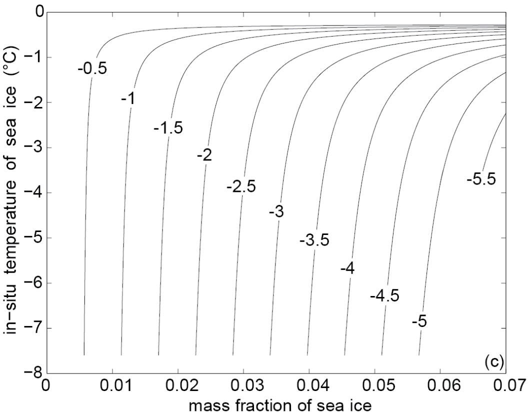
$$h^{\mathrm{seaice}} = \left(m_{\mathrm{Ih}}/m_{\mathrm{seaice}}\right)h^{\mathrm{Ih}} + \left(m_{\mathrm{brine}}/m_{\mathrm{seaice}}\right)h^{\mathrm{brine}},$$

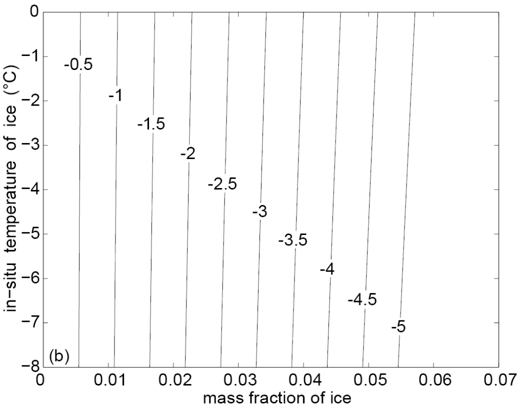
and the Absolute Salinity – enthalpy mixing diagram looks like





For sea ice melting into seawater at $p = 0$ dbar with initial properties $S_{\mathrm{A}}^{\mathrm{i}} = S_{\mathrm{SO}} = 35.16504\ \mathrm{g\ kg^{-1}}$, $\Theta^{\mathrm{i}} = 4°\mathrm{C}$, and with the sea ice salinity taken to be $S_{\mathrm{A}}^{\mathrm{seaice}} = 5\ \mathrm{g kg^{-1}}$, the change in Conservative Temperature is shown in the left-hand figure below. The right-hand figure shows the corresponding plot when $S_{\mathrm{A}}^{\mathrm{seaice}} = 0\ \mathrm{g kg^{-1}}$.



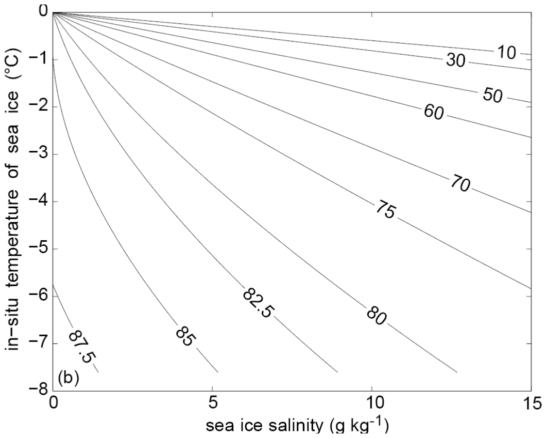


What is the ratio of the changes in Conservative Temperature and Absolute Salinity of seawater when a vanishingly small mass of sea ice melts into it? To find this ratio we again linearize the above expressions for a vanishingly small mass fraction of sea ice that melts, giving

$$\left(S_{\mathrm{A}} - S_{\mathrm{A}}^{\mathrm{seaice}}\right)\frac{\delta\Theta}{\delta S_{\mathrm{A}}}\bigg|_{\text{melting at constant p}} = \frac{\left(1-\dfrac{S_{\mathrm{A}}^{\mathrm{seaice}}}{S_{\mathrm{A}}^{\mathrm{brine}}}\right)\left(h-h^{\mathrm{Ih}}-S_{\mathrm{A}}\hat{h}_{S_{\mathrm{A}}}\right)+\dfrac{S_{\mathrm{A}}^{\mathrm{seaice}}}{S_{\mathrm{A}}^{\mathrm{brine}}}\left(h-h^{\mathrm{brine}}-\left[S_{\mathrm{A}}-S_{\mathrm{A}}^{\mathrm{brine}}\right]\hat{h}_{S_{\mathrm{A}}}\right)}{\hat{h}_{\Theta}}$$

and this is illustrated below at $p = 0$ dbar and at $S_{\mathrm{A}}^{\mathrm{i}} = S_{\mathrm{SO}} = 35.16504\ \mathrm{g\ kg^{-1}}$ and $\Theta^{\mathrm{i}} = 1°\mathrm{C}$.

$$\left(S_{\mathrm{A}} - S_{\mathrm{A}}^{\mathrm{seaice}}\right)\frac{\delta\Theta}{\delta S_{\mathrm{A}}}\bigg|_{\text{melting at } p\,=\,0}$$



Frazil ice formation

When seawater at the freezing temperature undergoes upwards vertical motion so that its pressure decreases, frazil forms, primarily due to the increase in the freezing temperature as a result of the reduction in pressure. When this mixture of seawater and frazil continues to rise to lower pressures (assisted by the buoyancy provided by the presence of the ice), the frazil crystals will experience a larger change in *in situ* temperature than does the seawater, simply because the adiabatic lapse rate of ice is much larger (ten times as large) than that of seawater (as we have found above).

We will here consider this situation under the assumption that the frazil and the seawater moves together, so ignoring the tendency of the frazil to rise faster than the seawater, driven by the buoyancy of the individual ice crystals. We further assume that the uplift rate is sufficiently small that the *in situ* temperature of the ice and the seawater are the same at each pressure, this temperature being the freezing temperature. Under these conditions no entropy is produced during the freezing process, i.e., this freezing process is reversible and can be reversed by increasing the pressure, leading to the related reversible ice melt.

We will study the thermodynamics of this process of adiabatic uplift of a seawater-ice mixture via a thought process composed of two separate steps (Fig. Ice_13). First we imagine the mixture of pre-existing ice and seawater to undergo a reduction in pressure *but without any exchange* of heat, water or salt between the two phases. That is, during this first part of the process the mass of ice and the mass of seawater remain constant, and the change in the enthalpy of the ice and the change in the enthalpy of the seawater are only due to the pressure change. During this adiabatic process an (infinitesimal) contrast in *in situ* temperature will develop between the ice phase and the seawater phase because the adiabatic lapse rate of ice is much larger (by about an order of magnitude) than that of seawater.

During the second part of our thought experiment the ice and seawater phases will be allowed to equilibrate their temperatures and further frazil ice will form so that the temperature of both the ice and seawater phases and the final Absolute Salinity of the seawater phase will be consistent with the freezing temperature at this pressure. This part of our thought experiment occurs at constant pressure and so, from the First Law of Thermodynamics, we know that enthalpy is conserved.

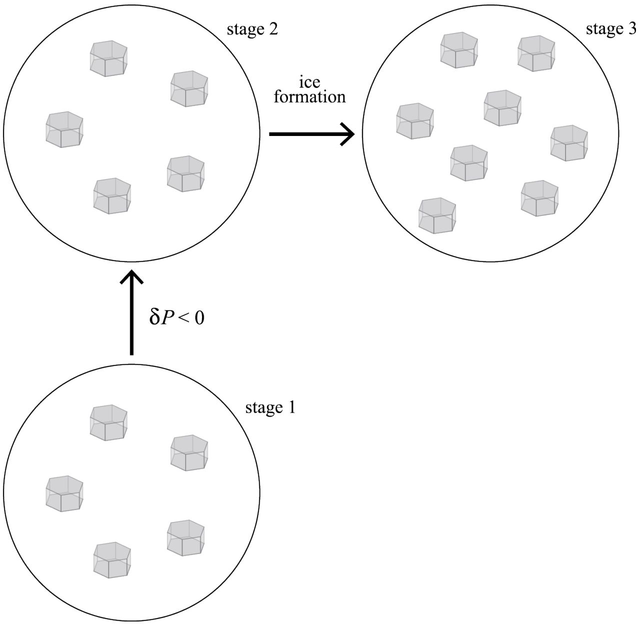


Figure 13. Sketch showing the two-step thought process involved with quantifying the formation of frazil ice Ih by the adiabatic uplift of a seawater parcel which may contain pre-existing frazil ice. The step from stage 1 to stage 2 is undertaken without any exchange of heat or mass between the seawater and ice Ih phases. While the *in situ* temperatures of the seawater and ice phases are assumed to be identical at stage 1, at stage 2 they are unequal because the adiabatic lapse rate of ice Ih is much larger than that of seawater. The step from stage 2 to stage 3 is undertaken at constant pressure. In this step further ice forms (as shown by the increase in number of the frazil ice crystals) and at the end of this step, the seawater and ice phases have the same *in situ* temperature, namely the freezing temperature appropriate to (i) that pressure and (ii) the final value of seawater salinity.

Let the mass fraction of ice be w^{Ih}; the mass fraction of seawater in the ice-seawater mixture is then $\left(1 - w^{\mathrm{Ih}}\right)$. The total enthalpy per unit mass of the ice-seawater mixture at stage 1 of Fig. 13 is the weighted sum of the specific enthalpies of the two phases, namely

$$\left(1 - w_1^{\mathrm{Ih}}\right)\hat{h}\left(S_{\mathrm{A}1},\Theta_1,p_1\right) \; + \; w_1^{\mathrm{Ih}}\tilde{h}^{\mathrm{Ih}}\left(\theta_1^{\mathrm{Ih}},p_1\right), \qquad \text{(Ice\_34)}$$

where we have chosen to write the specific enthalpy of ice in the functional form $\tilde{h}^{\mathrm{Ih}}\left(\theta^{\mathrm{Ih}},p\right)$ where the temperature variable is the potential temperature of ice θ^{Ih} with reference pressure 0 dbar (θ^{Ih} is not to be confused with the potential temperature of seawater θ, since these two potential temperatures are not equal).

In going from stage 1 to stage 2, both the seawater and ice phases undergo an adiabatic change of pressure δP which changes their specific enthalpies by $v\delta P$ and $v^{\mathrm{Ih}}\delta P$ respectively (here v and v^{Ih} are the specific volumes). Hence at stage 2 the total enthalpy per unit mass of the ice-seawater mixture is (noting that $w_2^{\mathrm{Ih}} = w_1^{\mathrm{Ih}}$ and that at leading order in the perturbation quantities, it is immaterial whether v and v^{Ih} are evaluated at the properties of stage 1 or those of stage 2)

$$\left(1-w_1^{\mathrm{Ih}}\right)\left[\hat{h}\left(S_{\mathrm{A}1},\Theta_1,p_1\right)+v\,\delta P\right] \;+\; w_1^{\mathrm{Ih}}\left[\tilde{h}^{\mathrm{Ih}}\left(\theta_1^{\mathrm{Ih}},p_1\right)+v^{\mathrm{Ih}}\,\delta P\right]. \qquad \text{(Ice\_35)}$$

In going from stage 2 to stage 3, the total enthalpy of the mixture is conserved. Hence we equate the total enthalpies at these two stages, giving

$$\begin{aligned}\left(1-w_1^{\mathrm{Ih}}\right)\left[\hat{h}\left(S_{\mathrm{A}1},\Theta_1,p_1\right)+v\,\delta P\right] \;+\; w_1^{\mathrm{Ih}}\left[\tilde{h}^{\mathrm{Ih}}\left(\theta_1^{\mathrm{Ih}},p_1\right)+v^{\mathrm{Ih}}\,\delta P\right] \;=&\\ \left(1-w_3^{\mathrm{Ih}}\right)\hat{h}\left(S_{\mathrm{A}3},\Theta_3,p_3\right) \;+\; w_3^{\mathrm{Ih}}\tilde{h}^{\mathrm{Ih}}\left(\theta_3^{\mathrm{Ih}},p_3\right).&\end{aligned} \qquad \text{(Ice\_36)}$$

For an externally-imposed change in pressure this equation may be regarded as giving the amount of new ice formed $w_3^{\mathrm{Ih}}-w_1^{\mathrm{Ih}}$ due to the adiabatic uplifting of the ice-seawater mixture. The other important constraint that we know is that the ice-seawater mixture is at the freezing temperature at both stages 1 and 3. This turns out to enough information to solve the problem.

The enthalpies $\hat{h}\left(S_{\mathrm{A}3},\Theta_3,p_3\right)$ and $\tilde{h}^{\mathrm{Ih}}\left(\theta_3^{\mathrm{Ih}},p_3\right)$ on the right-hand side of Eq. (Ice_36) are now expanded in a Taylor series about the values at stage 1, keeping the leading order terms. The pressure derivatives of these enthalpies, being the specific volumes of seawater and of ice, give terms that cancel with the corresponding terms on the left-hand side of the equation to leading order. The remaining leading-order terms are

$$\left(h-h^{\mathrm{Ih}}\right)\delta w^{\mathrm{Ih}} \;-\; \left(1-w^{\mathrm{Ih}}\right)\left(\hat{h}_{S_{\mathrm{A}}}\,\delta S_{\mathrm{A}}+\hat{h}_{\Theta}\,\delta\Theta\right) \;-\; w^{\mathrm{Ih}}\tilde{h}^{\mathrm{Ih}}_{\theta^{\mathrm{Ih}}}\,\delta\theta^{\mathrm{Ih}} \;=\; 0 \qquad \text{(Ice\_37)}$$

where $\delta w^{\mathrm{Ih}} = w_3^{\mathrm{Ih}}-w_1^{\mathrm{Ih}}$. Since the salt always resides in the seawater phase, the product $\left(1-w^{\mathrm{Ih}}\right)S_{\mathrm{A}}$ is constant so that

$$S_{\mathrm{A}}\,\delta w^{\mathrm{Ih}} \;=\; \left(1-w^{\mathrm{Ih}}\right)\delta S_{\mathrm{A}}\,, \qquad \text{(Ice\_38)}$$

which reduces Eqn. (Ice_37) to

$$\boxed{\left(h-h^{\mathrm{Ih}}-S_{\mathrm{A}}\hat{h}_{S_{\mathrm{A}}}\right)\delta S_{\mathrm{A}} \;-\; S_{\mathrm{A}}\hat{h}_{\Theta}\,\delta\Theta \;-\; S_{\mathrm{A}}\frac{w^{\mathrm{Ih}}}{\left(1-w^{\mathrm{Ih}}\right)}\tilde{h}^{\mathrm{Ih}}_{\theta^{\mathrm{Ih}}}\,\delta\theta^{\mathrm{Ih}} \;=\; 0}\,. \qquad \text{(Ice\_39)}$$

One of our key results for frazil ice is already apparent from this equation, namely that as the mass fraction of frazil ice w^{Ih} tends to zero, Eqn. (Ice_39) tends to our existing result Eqn. (Ice_16) for the ratio $\delta\Theta/\delta S_{\mathrm{A}}$ for the melting of ice Ih into seawater, repeated here,

$$\begin{aligned} S_{\mathrm{A}}\left.\frac{\delta\Theta}{\delta S_{\mathrm{A}}}\right|_{\text{melting at constant p}} &= \frac{h-h^{\mathrm{Ih}}-S_{\mathrm{A}}\hat{h}_{S_{\mathrm{A}}}}{\hat{h}_{\Theta}} \\ &= \frac{\hat{h}\left(S_{\mathrm{A}},\Theta,p\right)-h^{\mathrm{Ih}}\left(t^{\mathrm{Ih}},p\right)-S_{\mathrm{A}}\hat{h}_{S_{\mathrm{A}}}\left(S_{\mathrm{A}},\Theta,p\right)}{\hat{h}_{\Theta}\left(S_{\mathrm{A}},\Theta,p\right)}\,. \end{aligned} \qquad \textbf{(Ice\_16)}$$

However, there is an important difference as well, namely that the present frazil ice relation Eqn. (Ice_39) for the ratio $\delta\Theta/\delta S_{\mathrm{A}}$ is actually simpler (or more restrictive) than Eqn. (Ice_16) because the temperatures of both the ice and seawater components are constrained to be at the freezing temperature; the ice temperature cannot be lower that the freezing temperature nor can the Conservative Temperature of the seawater exceed its freezing temperature. Hence in the limit as the mass fraction of frazil ice w^{Ih} tends to zero, as the pressure of a seawater-frazil mixture is changed, the ratio $\delta\Theta/\delta S_{\mathrm{A}}$ is illustrated by the *equilibrium* situation of our existing Figures Ice_5(a) and (b). We will return to this; for now this paragraph is just a heads up on the comparison between what we have derived already (Eqn. (Ice_16)) and where we are headed with the equations for frazil.

Returning to the more general situation in which $w^{\rm Ih}$ is not vanishingly small, we need to evaluate $\tilde{h}^{\rm Ih}_{\theta^{\rm Ih}}\,\delta\theta^{\rm Ih}$ in terms of differentials of Absolute Salinity and pressure. The partial differential $\tilde{h}^{\rm Ih}_{\theta^{\rm Ih}}$ can be written as

$$\tilde{h}^{\rm Ih}_{\theta^{\rm Ih}} \;\equiv\; \frac{\partial h^{\rm Ih}}{\partial \theta^{\rm Ih}}\bigg|_p \;=\; \frac{\partial h^{\rm Ih}}{\partial t^{\rm Ih}}\bigg|_p \frac{\partial t^{\rm Ih}}{\partial \theta^{\rm Ih}}\bigg|_p \;=\; c_p^{\rm Ih}\,\frac{\partial t^{\rm Ih}}{\partial \theta^{\rm Ih}}\bigg|_p . \qquad \text{(Ice\_40)}$$

The *in situ* temperature of ice Ih can be expressed as a function of the potential temperature of ice Ih and pressure as $t^{\rm Ih} = t^{\rm Ih}\left(\theta^{\rm Ih},p\right)$ so that the total differential of the *in situ* temperature of ice is

$$dt^{\rm Ih} \;=\; \frac{\partial t^{\rm Ih}}{\partial \theta^{\rm Ih}}\bigg|_p \, d\theta^{\rm Ih} \;+\; \Gamma^{\rm Ih} dP . \qquad \text{(Ice\_41)}$$

This equation applies to any material differentials $dt^{\rm Ih}, d\theta^{\rm Ih}$ and dP, and in particular will apply to the differences between these properties at stage 1 and stage 3 of our thought process. Hence we can write

$$\delta t^{\rm Ih} \;=\; \frac{\partial t^{\rm Ih}}{\partial \theta^{\rm Ih}}\bigg|_p \, \delta\theta^{\rm Ih} \;+\; \Gamma^{\rm Ih}\delta P . \qquad \text{(Ice\_42)}$$

But the ice at both stages 1 and 3 is at the freezing temperature $t_{\rm freezing} = t_{\rm freezing}\left(S_{\rm A},p\right)$ so that $\delta t^{\rm Ih}$ can also be expressed as

$$\delta t^{\rm Ih} \;=\; \frac{\partial t_{\rm freezing}}{\partial S_{\rm A}}\bigg|_p \, \delta S_{\rm A} \;+\; \frac{\partial t_{\rm freezing}}{\partial P}\bigg|_{S_{\rm A}} \, \delta P , \qquad \text{(Ice\_43)}$$

and the partial derivatives here are known functions of the Gibbs functions of ice Ih and seawater.

Combining Eqns. (Ice_42) and (Ice_43) and using the result in Eqn. (Ice_40) gives our desired result for $\tilde{h}^{\rm Ih}_{\theta^{\rm Ih}}\,\delta\theta^{\rm Ih}$, namely

$$\tilde{h}^{\rm Ih}_{\theta^{\rm Ih}}\,\delta\theta^{\rm Ih} \;=\; c_p^{\rm Ih}\left[\frac{\partial t_{\rm freezing}}{\partial S_{\rm A}}\bigg|_p \, \delta S_{\rm A} \;+\; \left(\frac{\partial t_{\rm freezing}}{\partial P}\bigg|_{S_{\rm A}} \;-\; \Gamma^{\rm Ih}\right)\delta P\right]. \qquad \text{(Ice\_44)}$$

Substituting this equation into Eq. (39) gives a relationship between only $\delta S_{\rm A}, \delta\Theta$, and δP, namely

$$\left(h - h^{\rm Ih} - S_{\rm A}\hat{h}_{S_{\rm A}} \;-\; S_{\rm A}\frac{w^{\rm Ih}}{\left(1-w^{\rm Ih}\right)}c_p^{\rm Ih}\frac{\partial t_{\rm freezing}}{\partial S_{\rm A}}\bigg|_p\right)\delta S_{\rm A} \;-\; S_{\rm A}\hat{h}_{\Theta}\,\delta\Theta$$
$$\qquad\qquad \text{(Ice\_45)}$$
$$\;-\; S_{\rm A}\frac{w^{\rm Ih}}{\left(1-w^{\rm Ih}\right)}c_p^{\rm Ih}\left(\frac{\partial t_{\rm freezing}}{\partial P}\bigg|_{S_{\rm A}} \;-\; \Gamma^{\rm Ih}\right)\delta P \;=\; 0.$$

Another relationship between $\delta S_{\rm A}, \delta\Theta$, and δP can be found from the knowledge that in both stages 1 and 3 the seawater is at the freezing Conservative Temperature, and since $\Theta_{\rm freezing}$ is a function of only $S_{\rm A}$ and P, the differences $\delta S_{\rm A}, \delta\Theta$, and δP are related by

$$\delta\Theta \;=\; \frac{\partial \Theta_{\rm freezing}}{\partial S_{\rm A}}\bigg|_p \, \delta S_{\rm A} \;+\; \frac{\partial \Theta_{\rm freezing}}{\partial P}\bigg|_{S_{\rm A}} \, \delta P , \qquad \text{(Ice\_46)}$$

and expressions for these partial derivatives are known in terms of the Gibbs functions of seawater and of ice (we do not derive them here).

Eqns. (Ice_45) and (Ice_46) are two equations in $\delta S_{\rm A}$, $\delta\Theta$, and δP from which we can find our desired relations for the ratios of changes in our seawater-frazil ice mixture due to adiabatic uplift, namely $\delta\Theta/\delta S_{\rm A}$, $\delta\Theta/\delta P$ and

$\delta S_{\mathrm{A}}/\delta P$. By eliminating the pressure difference from these two equations we find that

$$S_{\mathrm{A}}\frac{\delta\Theta}{\delta S_{\mathrm{A}}}\bigg|_{\mathrm{frazil}} = \frac{\left(h - h^{\mathrm{Ih}} - S_{\mathrm{A}}\hat{h}_{S_{\mathrm{A}}} - S_{\mathrm{A}}\frac{w^{\mathrm{Ih}}}{\left(1-w^{\mathrm{Ih}}\right)}c_p^{\mathrm{Ih}}\left[\frac{\partial t_{\mathrm{freezing}}}{\partial S_{\mathrm{A}}}\bigg|_p - \frac{\left(\frac{\partial t_{\mathrm{freezing}}}{\partial P}\Big|_{S_{\mathrm{A}}} - \Gamma^{\mathrm{Ih}}\right)}{\frac{\partial \Theta_{\mathrm{freezing}}}{\partial P}\Big|_{S_{\mathrm{A}}}}\frac{\partial \Theta_{\mathrm{freezing}}}{\partial S_{\mathrm{A}}}\bigg|_p\right]\right)}{\left(\hat{h}_{\Theta} + \frac{w^{\mathrm{Ih}}}{\left(1-w^{\mathrm{Ih}}\right)}c_p^{\mathrm{Ih}}\frac{\left(\frac{\partial t_{\mathrm{freezing}}}{\partial P}\Big|_{S_{\mathrm{A}}} - \Gamma^{\mathrm{Ih}}\right)}{\frac{\partial \Theta_{\mathrm{freezing}}}{\partial P}\Big|_{S_{\mathrm{A}}}}\right)}. \qquad \text{(Ice\_47)}$$

The leading terms in both the numerator and denominator, namely $h - h^{\mathrm{Ih}} - S_{\mathrm{A}}\hat{h}_{S_{\mathrm{A}}}$ and $\hat{h}_{\Theta}$ are the same as in Eqn. (Ice_16) which applies to the melting of ice Ih into seawater at fixed pressure, the only difference being that in the present case ***both the ice and the seawater are at the freezing temperature***.

So, as the mass fraction of ice tends to zero, Eqn. (Ice_47) tends to Eqn. (Ice_16), so that at $w^{\mathrm{Ih}} = 0$ Eqn. (Ice_47) can be illustrated by Fig. Ice_5(a), which is repeated below.

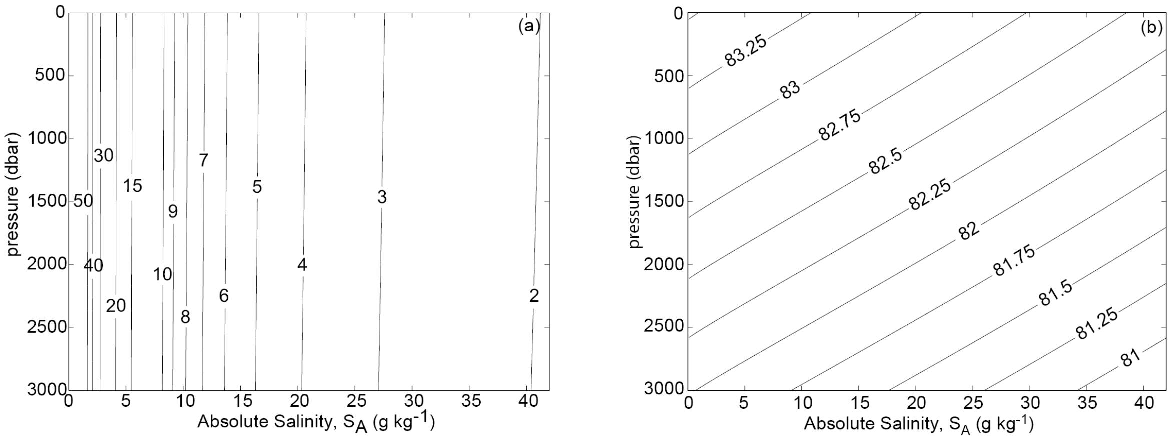


Figure Ice_5. (repeat of this figure) (a) The ratio of the change of Conservative Temperature to that of Absolute Salinity when the melting occurs *very near thermodynamic equilibrium* conditions, $\delta\Theta/\delta S_{\mathrm{A}}\big|_{\mathrm{equilibrium}}$, from Eqn. (Ice_16) with the seawater enthalpy evaluated at the freezing Conservative Temperature and with the ice enthalpy evaluated at the *in situ* freezing temperature, at each value of pressure and Absolute Salinity. The values contoured have units of $\mathrm{K}\left(\mathrm{g\ kg}^{-1}\right)^{-1}$.
(b) This panel is simply Absolute Salinity S_{A} times the values of panel (a), that is, it is the right-hand side of Eqn. (Ice_16), evaluated at equilibrium conditions.

For non-zero ice mass fraction Eqn. (Ice_47) is plotted in Fig. Ice_14(a) at $S_{\mathrm{A}} = S_{\mathrm{SO}} = 35.16504\ \mathrm{g\ kg}^{-1}$ (actually what is plotted is $\delta\Theta/\delta S_{\mathrm{A}}\big|_{\mathrm{frazil}}$). The dependence on the mass fraction of sea ice can be illustrated with the case $w^{\mathrm{Ih}} = 0.1$ when $\delta\Theta/\delta S_{\mathrm{A}}\big|_{\mathrm{frazil}}$ is different to the value at $w^{\mathrm{Ih}} = 0$ by about 7.4%. Most of this sensitivity to w^{Ih} comes from the denominator in Eqn. (Ice_47). Eqn. (Ice_47) is again illustrated in Fig. Ice_15(a) where we show contours of $S_{\mathrm{A}}\,\delta\Theta/\delta S_{\mathrm{A}}\big|_{\mathrm{frazil}}$ at the fixed salinity $S_{\mathrm{A}} = S_{\mathrm{SO}} = 35.16504\ \mathrm{g\ kg}^{-1}$. That is, Fig.

Ice_15(a) is simply $35.16504 \text{ g kg}^{-1}$ times Fig. Ice_14(a), so that the quantity contoured in Fig. Ice_15(a) is in temperature units.

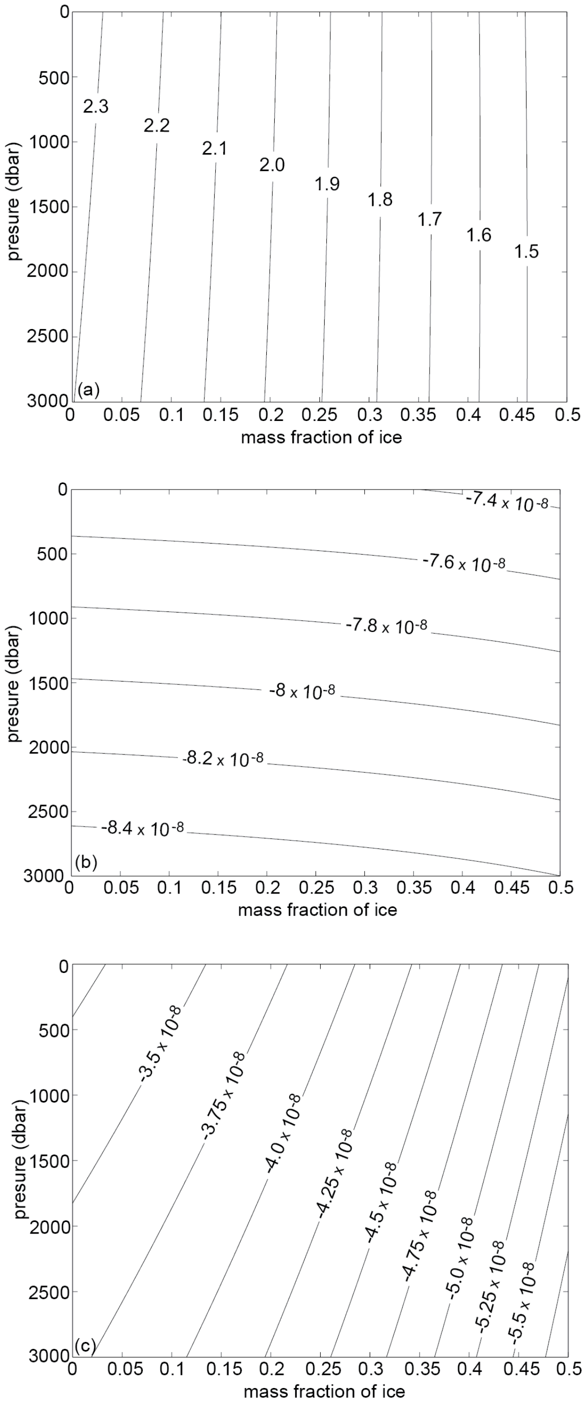


Figure Ice_14. (a) Plot of $\delta\Theta/\delta S_{\mathrm{A}}\big|_{\mathrm{frazil}}$ from Eqn. (Ice_47) as a function of the ice mass fraction w^{Ih} and pressure. (b) Plot of $\delta\Theta/\delta P\big|_{\mathrm{frazil}}$ from Eqn. (Ice_48) as a function of the ice mass fraction w^{Ih} and pressure. (c) Plot of $\delta S_{\mathrm{A}}/\delta P\big|_{\mathrm{frazil}}$ from Eqn. (Ice_49) as a function of the ice mass fraction w^{Ih} and pressure. All three panels have the seawater salinity $S_{\mathrm{A}} = S_{\mathrm{SO}} = 35.16504 \text{ g kg}^{-1}$. Panel (a) has units of $\mathrm{K}\left(\text{g kg}^{-1}\right)^{-1}$, panel (b) is in $\mathrm{K}\left(\mathrm{Pa}\right)^{-1}$, while panel (c) is in $\left(\text{g kg}^{-1}\right)\left(\mathrm{Pa}\right)^{-1}$. The values contoured in this figure were evaluated from the GSW algorithm **gsw_frazil_ratios_adiabatic** of the GSW Oceanographic Toolbox (www.TEOS-10.org).

Similarly, by eliminating δS_{A} from Eqns. (Ice_45) and (Ice_46) we find

$$\left.\frac{\delta \Theta}{\delta P}\right|_{\mathrm{frazil}} = \left.\frac{\partial \Theta_{\mathrm{freezing}}}{\partial P}\right|_{S_{\mathrm{A}}} \frac{\left(h - h^{\mathrm{Ih}} - S_{\mathrm{A}} \hat{h}_{S_{\mathrm{A}}} - S_{\mathrm{A}} \frac{w^{\mathrm{Ih}}}{\left(1 - w^{\mathrm{Ih}}\right)} c_p^{\mathrm{Ih}} \left[\left.\frac{\partial t_{\mathrm{freezing}}}{\partial S_{\mathrm{A}}}\right|_p - \frac{\left(\left.\frac{\partial t_{\mathrm{freezing}}}{\partial P}\right|_{S_{\mathrm{A}}} - \Gamma^{\mathrm{Ih}} \right)}{\left.\frac{\partial \Theta_{\mathrm{freezing}}}{\partial P}\right|_{S_{\mathrm{A}}}} \left.\frac{\partial \Theta_{\mathrm{freezing}}}{\partial S_{\mathrm{A}}}\right|_p \right] \right)}{\left(h - h^{\mathrm{Ih}} - S_{\mathrm{A}} \hat{h}_{S_{\mathrm{A}}} - S_{\mathrm{A}} \hat{h}_{\Theta} \left.\frac{\partial \Theta_{\mathrm{freezing}}}{\partial S_{\mathrm{A}}}\right|_p - S_{\mathrm{A}} \frac{w^{\mathrm{Ih}}}{\left(1 - w^{\mathrm{Ih}}\right)} c_p^{\mathrm{Ih}} \left.\frac{\partial t_{\mathrm{freezing}}}{\partial S_{\mathrm{A}}}\right|_p \right)}, \qquad \text{(Ice\_48)}$$

and when $\delta\Theta$ is eliminated from these same two equations we find

$$\frac{1}{S_{\mathrm{A}}} \left.\frac{\delta S_{\mathrm{A}}}{\delta P}\right|_{\mathrm{frazil}} = \left.\frac{\partial \Theta_{\mathrm{freezing}}}{\partial P}\right|_{S_{\mathrm{A}}} \frac{\left(\hat{h}_{\Theta} + \frac{w^{\mathrm{Ih}}}{\left(1 - w^{\mathrm{Ih}}\right)} c_p^{\mathrm{Ih}} \frac{\left(\left.\frac{\partial t_{\mathrm{freezing}}}{\partial P}\right|_{S_{\mathrm{A}}} - \Gamma^{\mathrm{Ih}} \right)}{\left.\frac{\partial \Theta_{\mathrm{freezing}}}{\partial P}\right|_{S_{\mathrm{A}}}} \right)}{\left(h - h^{\mathrm{Ih}} - S_{\mathrm{A}} \hat{h}_{S_{\mathrm{A}}} - S_{\mathrm{A}} \hat{h}_{\Theta} \left.\frac{\partial \Theta_{\mathrm{freezing}}}{\partial S_{\mathrm{A}}}\right|_p - S_{\mathrm{A}} \frac{w^{\mathrm{Ih}}}{\left(1 - w^{\mathrm{Ih}}\right)} c_p^{\mathrm{Ih}} \left.\frac{\partial t_{\mathrm{freezing}}}{\partial S_{\mathrm{A}}}\right|_p \right)}. \qquad \text{(Ice\_49)}$$

The variation of Conservative Temperature with pressure under frazil ice conditions, $\delta\Theta/\delta P\big|_{\mathrm{frazil}}$, from Eqn. (Ice_48) is plotted in Fig. Ice_14(b) at $S_{\mathrm{A}} = S_{\mathrm{SO}} = 35.16504\ \mathrm{g\ kg}^{-1}$. It is seen that $\delta\Theta/\delta P\big|_{\mathrm{frazil}}$ is quite insensitive to the frazil ice mass fraction w^{Ih}. This is confirmed in Fig. Ice_15(c) where we show the difference between $\delta\Theta/\delta P\big|_{\mathrm{frazil}}$ and the corresponding derivative of $\Theta_{\mathrm{freezing}}$ with pressure *at constant Absolute Salinity*, $\partial\Theta_{\mathrm{freezing}}/\partial P\big|_{S_{\mathrm{A}}}$.

The variation of Absolute Salinity with pressure under frazil ice conditions, $\delta S_{\mathrm{A}}/\delta P\big|_{\mathrm{frazil}}$, from Eqn. (Ice_49) is plotted in Figure Ice_14(c) at $S_{\mathrm{A}} = S_{\mathrm{SO}} = 35.16504\ \mathrm{g\ kg}^{-1}$. This figure follows, of course, as simply the ratio of the figures of panels (a) and (b).

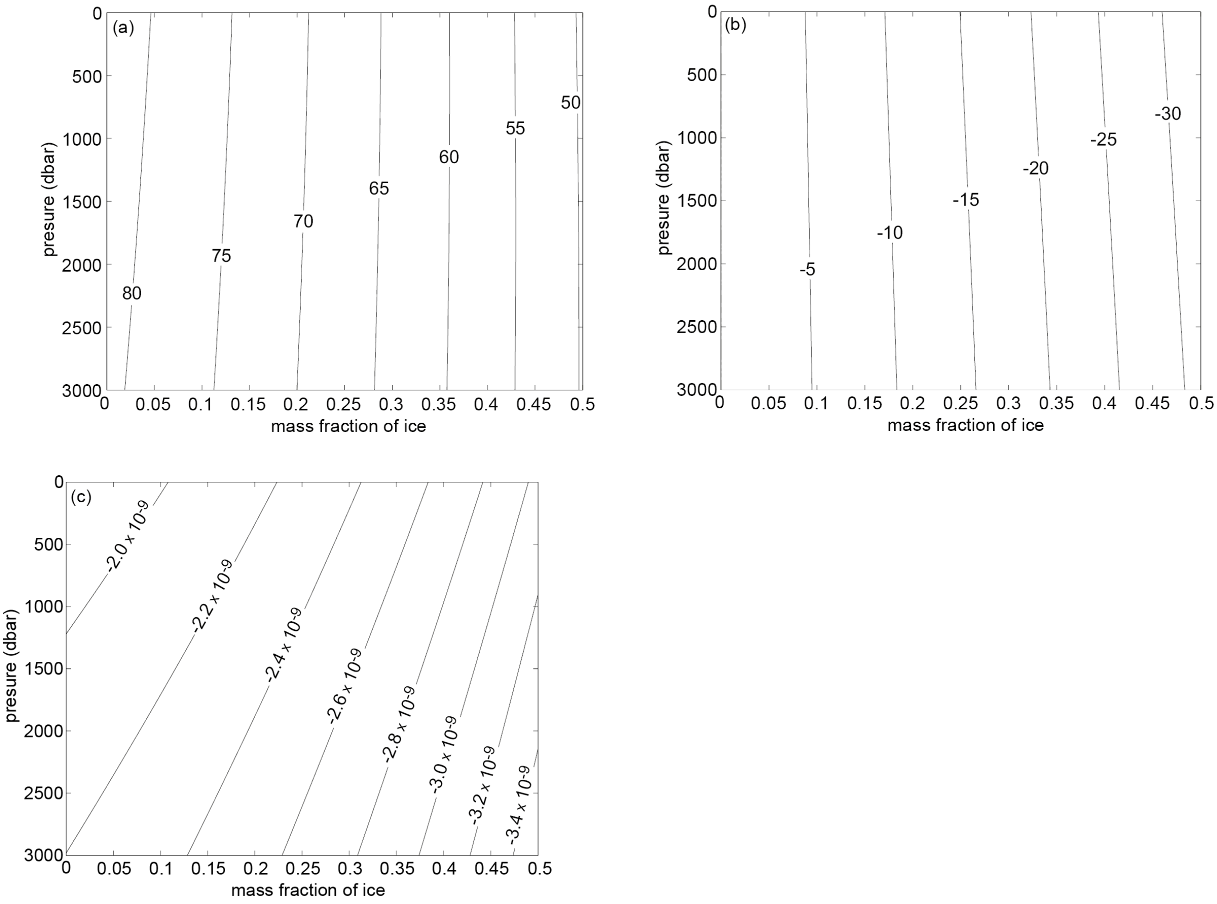


Figure Ice_15. (a) Plot of $S_{\mathrm{A}}\,\delta\Theta/\delta S_{\mathrm{A}}\big|_{\mathrm{frazil}}$ from Eqn. (Ice_47) as a function of the ice mass fraction w^{Ih} and pressure, for $S_{\mathrm{A}} = S_{\mathrm{SO}} = 35.16504\ \mathrm{g\ kg^{-1}}$. This is simply $35.16504\ \mathrm{g\ kg^{-1}}$ times Figure Ice_14(a).

(b) Contour plot of panel (a) with the values of $S_{\mathrm{A}}\,\delta\Theta/\delta S_{\mathrm{A}}\big|_{\mathrm{frazil}}$ evaluated at ice mass fraction $w^{\mathrm{Ih}} = 0$ subtracted at each pressure.

(c) The difference between $\delta\Theta/\delta P\big|_{\mathrm{frazil}}$ and the corresponding derivative of $\Theta_{\mathrm{freezing}}$ with pressure *at constant Absolute Salinity*, $\partial\Theta_{\mathrm{freezing}}/\partial P\big|_{S_{\mathrm{A}}}$ (obtained from **gsw_CT_freezing_first_derivatives**). The contoured values of panel (c) are in $\mathrm{K}\left(\mathrm{Pa}\right)^{-1}$ and the seawater salinity was taken to be $S_{\mathrm{A}} = S_{\mathrm{SO}} = 35.16504\ \mathrm{g\ kg^{-1}}$. Notice that the numbers contoured here are only a few percent of those of $\delta\Theta/\delta P\big|_{\mathrm{frazil}}$ shown in Fig. Ice_14(b).

The dependence of $S_{\mathrm{A}}\,\delta\Theta/\delta S_{\mathrm{A}}\big|_{\mathrm{frazil}}$ on the mass fraction of ice is illustrated in Fig. Ice_15(b) which shows the difference relative to the case when $w^{\mathrm{Ih}} = 0$.

When no frazil is present in seawater, its Conservative Temperature is unaffected by adiabatic and isohaline changes in pressure, but its *in situ* temperature changes with pressure according to the adiabatic lapse rate Γ which is usually positive. When frazil is present in seawater, an increase in pressure results in changes in Conservative Temperature as contoured in Fig. Ice_14(b). This dependence of the temperature (both Conservative Temperature and *in situ* temperature) of the frazil-seawater mixture to changes in pressure is rather large and negative compared with the (usually positive) adiabatic lapse rate of seawater which is typically less than one twentieth of the values shown in Fig. Ice_14(b) for $\delta\Theta/\delta P\big|_{\text{frazil}}$, and is usually of the opposite sign. Another way of stating this is that the adiabatic lapse rate of the frazil-seawater mixture is large and negative when frazil is present, compared with the small and positive adiabatic lapse rate of seawater in the absence of frazil.

Note that the rate at which the freezing Conservative Temperature changes with Absolute Salinity *at fixed pressure*, $\partial\Theta_{\text{freezing}}/\partial S_{\text{A}}\big|_{p}$, is quite different (even different signs) to the corresponding change involving frazil ice as the pressure varies, $\delta\Theta/\delta S_{\text{A}}\big|_{\text{frazil}}$. A typical value of $\partial\Theta_{\text{freezing}}/\partial S_{\text{A}}\big|_{p}$ is $-0.0583\,\text{K}\,\text{g}^{-1}\,\text{kg}$ while a typical value of $\delta\Theta/\delta S_{\text{A}}\big|_{\text{frazil}}$ is $2.3\,\text{K}\,\text{g}^{-1}\,\text{kg}$. By contrast, we have seen that the variation of Conservative Temperature with pressure for frazil ice, $\delta\Theta/\delta P\big|_{\text{frazil}}$, is only a few percent different to the corresponding change at constant Absolute Salinity, $\partial\Theta_{\text{freezing}}/\partial P\big|_{S_{\text{A}}}$.

All of the material above in these lectures concerning ice, sea ice, and frazil ice can be found in the paper

> McDougall, T. J., P. M. Barker, R. Feistel and B. K. Galton-Fenzi, 2014: Melting of ice and sea ice into seawater, and frazil ice formation. *Journal of Physical Oceanography*, **44**, 1751-1775.

The properties of ice and its equilibrium properties with seawater can be evaluated using the GSW Oceanographic Toolbox, available from www.TEOS-10.org.

The interaction between ice and seawater in ocean models

In a model, the mixture of seawater and frazil ice must be advected and diffused, and the frazil crystals must be allowed to migrate vertically (Stokes drift), but two questions arise.

1 What variables should the model carry to conserve salt and "heat"?

2 How should thermodynamic equilibrium be re-established after the advection, diffusion and frazil-Stokes-drift part of each time step?

Bulk Absolute Salinity and Bulk potential enthalpy

We have shown that conserving the potential enthalpy of ice Ih is sufficiently accurate (rather than having to conserve the enthalpy of ice Ih during melting/freezing and then worry about how the enthalpy of ice Ih varies with pressure).

This greatly simplifies our task because the First Law of Thermodynamics can be simplified to be the conservation of the potential enthalpy of the seawater-ice mixture.

Along with the ice mass fraction, w^{Ih}, the conserved model variables during the advection and diffusion part of the time step should then be the "Bulk Absolute Salinity" $S_{\mathrm{A}}^{\mathrm{B}}$ and the "Bulk potential enthalpy", $\hbar^{\mathrm{B}}$,

$$S_{\mathrm{A}}^{\mathrm{B}} \equiv \left(1-w^{\mathrm{Ih}}\right)S_{\mathrm{A}} \qquad \text{and} \qquad \hbar^{\mathrm{B}} \equiv \left(1-w^{\mathrm{Ih}}\right)c_p^0\Theta + w^{\mathrm{Ih}}\hbar^{\mathrm{Ih}}\,.$$

After the advection, diffusion and frazil-Stokes-drift part of the time step, we have values of $S_{\mathrm{A1}}^{\mathrm{B}}$ and $\hbar_1^{\mathrm{B}}$, for the model box, but these values will not be in thermodynamic equilibrium with each other.

How do we re-establish thermodynamic equilibrium in the second half of the time step?

The thermodynamic equilibrium condition between seawater and frazil ice

During the equilibration process, there is no exchange of mass, salt or heat with neighbouring boxes, so the Bulk Absolute Salinity and the Bulk potential enthalpy are conserved, so that when thermodynamic equilibrium is reached at the end of the full time step we must have

$$\left(1-w^{\mathrm{Ih}}\right)S_{\mathrm{A}} = S_{\mathrm{A1}}^{\mathrm{B}} \qquad \text{and} \qquad \left(1-w^{\mathrm{Ih}}\right)c_p^0\Theta + w^{\mathrm{Ih}}\hbar^{\mathrm{Ih}} = \hbar_1^{\mathrm{B}},$$

and in addition, the values of all these thermodynamic variables mutually adjust so that they satisfy the **freezing condition** at the end of the full time step.

Hence we seek the zero of the function of the ice mass fraction w^{Ih}

$$0 = f\left(w^{\mathrm{Ih}}\right) = \hbar_1^{\mathrm{B}} - \left(1-w^{\mathrm{Ih}}\right)c_p^0\Theta_{\mathrm{freezing}}\left(S_{\mathrm{A}},p\right) - w^{\mathrm{Ih}}\,\hbar_{\mathrm{freezing}}^{\mathrm{Ih}}\left(S_{\mathrm{A}},p\right).$$

where S_{A} is related to w^{Ih} by $S_{\mathrm{A}} = S_{\mathrm{A1}}^{\mathrm{B}}\big/\left(1-w^{\mathrm{Ih}}\right)$. Note that $f\left(w^{\mathrm{Ih}}\right)$ is indeed a function of only the ice mass fraction w^{Ih}.

To solve $f\left(w^{\mathrm{Ih}}\right)=0$ we use an improved version of Newton's Method which converges at the rate $1+\sqrt{2}$ and is described in McDougall T. J. and S. J. Wotherspoon, 2014: A simple modification of Newton's method to achieve convergence of order $1+\sqrt{2}$. *Applied Mathematics Letters*, **29**, 20-25. http://dx.doi.org/10.1016/j.aml.2013.10.008 .

The derivative $f'\left(w^{\mathrm{Ih}}\right)$ is given by

$$f'\left(w^{\mathrm{Ih}}\right) \;=\; c_p^0\Theta_{\mathrm{freezing}}\left(S_{\mathrm{A}},p\right) \;-\; \hbar_{\mathrm{freezing}}^{\mathrm{Ih}}\left(S_{\mathrm{A}},p\right) \;-\; S_{\mathrm{A}}\,c_p^0\left.\frac{\partial\Theta_{\mathrm{freezing}}}{\partial S_{\mathrm{A}}}\right|_p$$

$$-\;\frac{S_{\mathrm{A}}\,w^{\mathrm{Ih}}}{\left(1-w^{\mathrm{Ih}}\right)}\left.\frac{\partial\hbar_{\mathrm{freezing}}^{\mathrm{Ih}}}{\partial S_{\mathrm{A}}}\right|_p\,.$$

An efficient way of determining when there is no frazil ice component

If, at the end of the first part of the time step $\hbar_1^{\mathrm{B}}$ is sufficiently "warm", there will be no frazil ice. In this case the solution is

$$w^{\mathrm{Ih}} \;=\; 0\,, \qquad S_{\mathrm{A}} \;=\; S_{\mathrm{A1}}^{\mathrm{B}} \qquad \text{and} \qquad \Theta \;=\; \hbar_1^{\mathrm{B}}\big/c_p^0\,.$$

What is an efficient way of detecting when $\hbar_1^{\mathrm{B}}$ is "too warm" for frazil ice to be present? Based on the definition of the equilibrium condition (repeated from above),

$$0 \;=\; f\left(w^{\mathrm{Ih}}\right) \;=\; \hbar_1^{\mathrm{B}}-\left(1-w^{\mathrm{Ih}}\right)c_p^0\Theta_{\mathrm{freezing}}\left(S_{\mathrm{A}},p\right) \;-\; w^{\mathrm{Ih}}\,\hbar_{\mathrm{freezing}}^{\mathrm{Ih}}\left(S_{\mathrm{A}},p\right),$$

we evaluate this function when the ice mass fraction is zero, that is,

$$f\left(w^{\mathrm{Ih}}=0\right) \;=\; \hbar_1^{\mathrm{B}}-\,c_p^0\Theta_{\mathrm{freezing}}\left(S_{\mathrm{A1}}^{\mathrm{B}},p\right),$$

and if this is positive then the answer is simply seawater (and no frazil ice), so that we can set, $w^{\mathrm{Ih}} \;=\; 0$, $S_{\mathrm{A}} \;=\; S_{\mathrm{A1}}^{\mathrm{B}}$ and $\Theta \;=\; \hbar_1^{\mathrm{B}}\big/c_p^0$. The computer time involved with making this decision is simply the time it takes to evaluate the freezing Conservative Temperature $\Theta_{\mathrm{freezing}}\left(S_{\mathrm{A1}},p\right)$.

The computer code needed to restore the seawater/ice properties at the end of the first half of the time step $\left(S_{\mathrm{A1}}^{\mathrm{B}},\hbar_1^{\mathrm{B}},p\right)$ to thermodynamic equilibrium is contained in the computationally efficient code

gsw_frazil_properties_potential_poly(SA_bulk, h_pot_bulk, p)

which has outputs of $\left(S_{\mathrm{A}},\Theta,w^{\mathrm{Ih}}\right)$.

Buoyancy frequency N

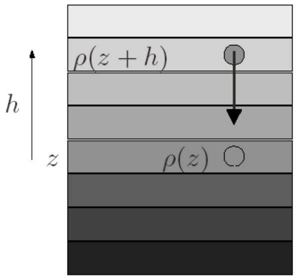


Figure 11-1 When an incompressible fluid parcel of density $\rho(z)$ is vertically displaced from level z to level $z+h$ in a stratified environment, a buoyancy force appears because of the density difference $\rho(z)-\rho(z+h)$ between the particle and the ambient fluid.

First consider the incompressible situation as illustrated in the figure. The figure shows that when a parcel is displaced upwards from its resting position in a stably stratified fluid, it experiences a downwards buoyant force because it is denser than the fluid of the environment that surrounds it at its new location. This force is indicated by the downwards-directed arrow in the figure.

When the fluid is compressible there is a vertical gradient of *in situ* density ρ, given by

$$\left.\frac{\partial\rho}{\partial P}\right|_{S_{\mathrm{A}},\Theta} P_z \;=\; \rho\kappa P_z\,, \qquad \text{(vertical isentropic density gradient)}$$

even when a fluid layer is completely well mixed so that Absolute Salinity, entropy and Conservative Temperature are all independent of height. In this compressible well-mixed case, the fluid parcel illustrated above would decrease its *in situ* density in moving upwards by the distance h, but at its new location, its density would be the same as that of the fluid around it at this height. So in order to quantify the vertical stability, that is, in order to quantify the vertical buoyant force that the parcel experiences at its new location, we need to take into account this vertical gradient of *in situ* density ρ due to the fluid's isentropic (and isohaline) compressibility κ.

The square of the buoyancy frequency (sometimes called the Brunt-Väisälä frequency), N^2, is given in terms of the vertical gradients of density and pressure, or in terms of the vertical gradients of Conservative Temperature and Absolute Salinity by (the g on the left-hand side is the gravitational acceleration, and x, y and z are the spatial Cartesian coordinates)

$$\begin{aligned} g^{-1}N^2 \;&=\; -\rho^{-1}\rho_z + \kappa P_z \;=\; -\rho^{-1}\left(\rho_z - P_z/c^2\right) \\ &=\; \alpha^{\Theta}\,\Theta_z\big|_{x,y} - \beta^{\Theta}\,\partial S_{\mathrm{A}}/\partial z\big|_{x,y}\,. \end{aligned} \qquad (3.10.1)$$

The buoyancy frequency N has units of radians per second, and since a radian is unitless, N has dimensions of s^{-1}. The buoyancy frequency N is the highest frequency of internal gravity waves in a density-stratified fluid like the ocean or atmosphere. The corresponding shortest period of internal gravity waves is $2\pi/N$ which varies from about 20 minutes in the upper ocean to a few hours in the deep ocean. (This is to be compared with $2\pi/f \geq 12$ hours where $f = 2\Omega\sin\phi = 1.458\,423\,00 x 10^{-4}\sin\phi\ \mathrm{s}^{-1}$, is the Coriolis parameter where ϕ is latitude and Ω is the rotation rate of the earth [in radians per second]).

For two seawater parcels separated by a small distance Δz in the vertical, an equally accurate method of calculating the buoyancy frequency is to bring both seawater parcels adiabatically and without exchange of matter to the average pressure and to calculate the difference in density of the two parcels after this change in pressure. In this way the potential densities, defined with reference pressure being the mean pressure of the two fluid parcels, are being compared at the same pressure. This common procedure calculates the buoyancy frequency N according to

$$N^2 = g\left(\alpha^\Theta \Theta_z - \beta^\Theta S_{\mathrm{A}_z}\right) \approx -\frac{g}{\rho}\frac{\Delta\rho^\Theta}{\Delta z}\,, \tag{3.10.2a}$$

or

$$N^2 = g^2\rho\left(\beta^\Theta S_{\mathrm{A}_P} - \alpha^\Theta \Theta_P\right) \approx \frac{g^2\,\Delta\rho^\Theta}{\Delta P}\,, \tag{3.10.2b}$$

where $\Delta\rho^\Theta$ is the difference between the potential densities of the two seawater parcels with the reference pressure being the average of the two original pressures of the seawater parcels. Eqn. (3.10.2b) has made use of the hydrostatic relation $P_z = -g\rho$, and ΔP is the difference in the pressures of the two parcels, in Pa.

This difference in potential density, $\Delta\rho^\Theta$, between two seawater parcels can be evaluated more easily when density is expressed in the form $\rho = \hat{\rho}\left(S_\mathrm{A}, \Theta, p\right)$ than when it is expressed in the form $\rho = \rho\left(S_\mathrm{A}, t, p\right)$; witness

$$\begin{aligned}\Delta\rho^\Theta &= \hat{\rho}\left(S_\mathrm{A}^\mathrm{deep}, \Theta^\mathrm{deep}, \bar{p}\right) - \hat{\rho}\left(S_\mathrm{A}^\mathrm{shallow}, \Theta^\mathrm{shallow}, \bar{p}\right)\\ &= \rho\left(S_\mathrm{A}^\mathrm{deep}, \theta\left(S_\mathrm{A}^\mathrm{deep}, t^\mathrm{deep}, p^\mathrm{deep}, \bar{p}\right), \bar{p}\right) - \rho\left(S_\mathrm{A}^\mathrm{shallow}, \theta\left(S_\mathrm{A}^\mathrm{shallow}, t^\mathrm{shallow}, p^\mathrm{shallow}, \bar{p}\right), \bar{p}\right)\end{aligned}$$

where $\bar{p} = \tfrac{1}{2}\left(p^\mathrm{deep} + p^\mathrm{shallow}\right)$. Compared with the first line of the above equation, the second line requires more calculations, and the expression is unnecessarily convoluted.

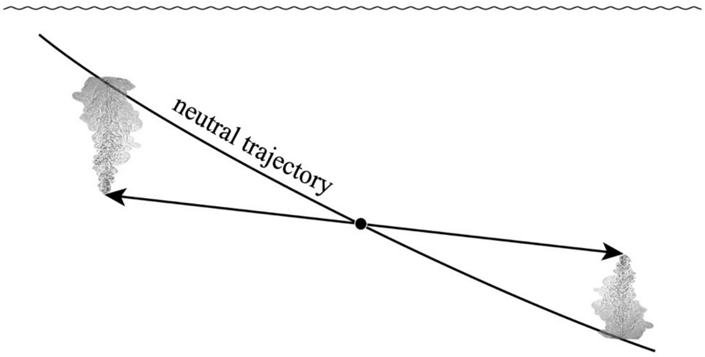
The "Stability Ratio" R_ρ of a vertical water column is defined as

$$R_\rho = \frac{\alpha^\Theta \Theta_z}{\beta^\Theta \left(S_\mathrm{A}\right)_z}\,. \tag{3.15.1}$$

R_ρ is the ratio of the vertical contribution from Conservative Temperature to that from Absolute Salinity to the static stability N^2 of the water column.

The neutral tangent plane

The "neutral tangent plane" is that plane in physical space in which the local parcel of seawater can be moved an infinitesimal distance without being subject to a vertical buoyant restoring force; it is the plane of neutral- or zero- buoyancy.



Take the seawater parcel at the central point and enclose it in an insulating plastic bag, then move it to a new location a small distance away. Its density will change by $\delta\rho = \rho\kappa\delta P$. At the same location the seawater environment has a density difference of $\delta\rho = \rho\left(\kappa\delta P + \beta^\Theta\delta S_\mathrm{A} - \alpha^\Theta\delta\Theta\right)$. If the seawater parcel is happy to sit still at its new location, it must not be feeling a vertical buoyant (Archimedean) force, and this requires that its density is equal to that of the environment at its new location. That is, we must have

$$\rho\kappa\delta P = \rho\left(\kappa\delta P + \beta^\Theta\delta S_\mathrm{A} - \alpha^\Theta\delta\Theta\right). \tag{Neutral\_1}$$

Hence, along a neutral trajectory the variations of S_A and Θ of the ocean obey

$$\beta^{\Theta}\delta S_{\mathrm{A}} = \alpha^{\Theta}\delta\Theta \ . \qquad \text{(Neutral\_2)}$$

Incidentally, this thought experiment involving the adiabatic and isohaline displacement of fluid parcels is typical of our thinking about turbulent fluxes. We imagine the adiabatic and isohaline movement of fluid parcels, and then we let these parcels mix molecularly with their surroundings. Central to this way of thinking about turbulent fluxes are the following two desirable properties of the tracer that is being mixed.

(1) it must be a "potential" property, for otherwise its value will change during the adiabatic and isohaline displacement so it is difficult to define a flux of the quantity, and

(2) it should preferably be a "conservative" fluid property so that when it does mix intimately (that is, molecularly) with its surrounding, we can be sure that no funny business is going on; no magic, undesirable production or destruction of the property.

Expressing this definition of a neutral tangent plane $\beta^{\Theta}\delta S_{\mathrm{A}} = \alpha^{\Theta}\delta\Theta$ in terms of the two-dimensional gradient of properties in the neutral tangent plane, we have that

$$-\rho^{-1}\nabla_{n}\rho + \kappa\nabla_{n}P \;=\; -\rho^{-1}\left(\nabla_{n}\rho - \nabla_{n}P/c^{2}\right) \;=\; \alpha^{\Theta}\nabla_{n}\Theta - \beta^{\Theta}\nabla_{n}S_{\mathrm{A}} \;=\; \mathbf{0}, \qquad (3.11.2)$$

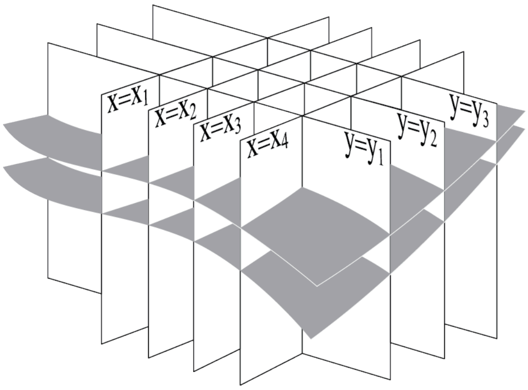
where, by way of reminder the relevant thermal expansion coefficient α^{Θ} and saline contraction coefficient β^{Θ} are defined by

$$\alpha^{\Theta} = -\frac{1}{\rho}\frac{\partial\rho}{\partial\Theta}\bigg|_{S_{\mathrm{A}},p} \qquad \text{and} \qquad \beta^{\Theta} = \frac{1}{\rho}\frac{\partial\rho}{\partial S_{\mathrm{A}}}\bigg|_{\Theta,p} . \qquad (2.18.3),\ (2.19.3)$$

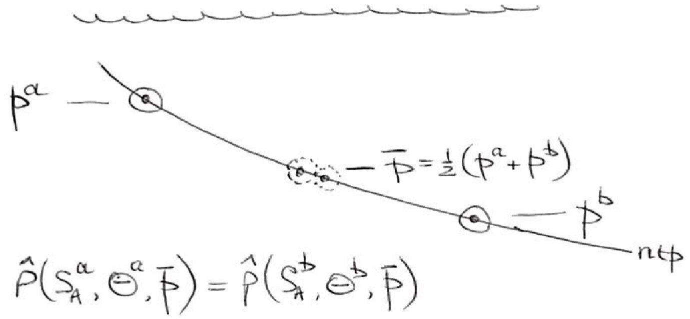
Here ∇_{n} is an example of a projected non-orthogonal gradient

$$\nabla_{r}\tau \equiv \frac{\partial\tau}{\partial x}\bigg|_{r}\mathbf{i} + \frac{\partial\tau}{\partial y}\bigg|_{r}\mathbf{j} + 0\mathbf{k}, \qquad (3.11.3)$$

that is widely used in oceanic and atmospheric theory and modelling. Horizontal distances are measured between the vertical planes of constant latitude x and longitude y while the values of the property τ are evaluated on the r surface (e. g. an isopycnal surface, or in the case of ∇_{n}, a neutral tangent plane). Note that $\nabla_{r}\tau$ has no vertical component; it is not directed along the r surface, but rather it points in exactly the horizontal direction.



A very accurate finite amplitude version of achieving $\beta^{\Theta}\delta S_{\mathrm{A}} = \alpha^{\Theta}\delta\Theta$ is to equate the potential densities of the two fluid parcels, each referenced to the average pressure $\bar{p} = 0.5\left(p^{\mathrm{a}} + p^{\mathrm{b}}\right)$. In this way, when two parcels, parcels a and b, are on a neutral tangent plane then $\hat{\rho}\left(S_{\mathrm{A}}^{\mathrm{a}},\Theta^{\mathrm{a}},\bar{p}\right) = \hat{\rho}\left(S_{\mathrm{A}}^{\mathrm{b}},\Theta^{\mathrm{b}},\bar{p}\right)$; see the figure below which involves the thought process of moving both parcels to pressure $\bar{p}$.



The (three dimensional) normal vector to the neutral tangent plane $\mathbf{n}$ is given by

$$\begin{aligned} g^{-1}N^{2}\,\mathbf{n} \;&=\; -\rho^{-1}\nabla\rho + \kappa\nabla P \;=\; -\rho^{-1}\left(\nabla\rho - \nabla P / c^{2}\right) \\ &=\; \alpha^{\Theta}\nabla\Theta - \beta^{\Theta}\,\nabla S_{\mathrm{A}}. \end{aligned} \tag{3.11.1}$$

As defined, $\mathbf{n}$ is not quite a unit normal vector, rather its vertical component is exactly $\mathbf{k}$, that is, its vertical component is unity ($\mathbf{k}\cdot\mathbf{n} = 1$).